THE MATHEMATICIAN.

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DISSERTATIONS

ON THE

RISE, PROGRESS, and IMPROVEMENT

OF

GEOMETRY.

ALSO,

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THE

Mathematician.

A DISSERTATION *upon the* ORIGIN, PROGRESS, *and* IMPROVEMENT *of* GEOMETRY.

AMONG those Arguments produced against the Opinion of *Aristotle*, to prove that the World was not eternal, but had a Beginning; that which is drawn from the late Invention of Arts and Sciences, seems to be of great Weight, and almost conclusive; for not only these, but the necessary Affairs of Life, such as Agriculture, Building of Houses, &c. had their Beginning within these 4000 Years, or the Compass of History.

What Soil first produced each Science is not quite clear, for the same Discoveries have appeared in different Parts of the World, without their

their having had any Communication with one another; for Instance, the Use of the Loadstone, the Invention of Printing, and Gunpowder, were discovered by the *Chinese* as well as by the *Europeans*.

As to the Methematical Sciences, it seems that the Preference therein should be given to the *Europeans*; and 'tis generally allowed that the *Chaldeans* were first possessed of them, especially Astronomy, which must imply Geometry. Whether *Abraham* taught these Sciences first to the *Egyptians*, when he went from *Ur* of the *Chaldees*, as some learned Men assert, is not clear; but on this we may depend, that the *Egyptians* were the first People that cultivated Geometry, being compelled thereto by Necessity, the Mother of Inventions, in order to ascertain to every Man his legal Property and Estate, in a Country where Boundaries and Land-Marks were swept away and confounded by yearly Inundations *.

That the *Egyptians*, in their antient, free, monarchical State, were acquainted with some of the simple Elements and easy Problems in Geometry, is not denied; but we cannot believe they made any great Improvements in the abstruse Parts thereof, since to *Pythagoras* (the famous Philosopher of *Samos*, who flourished so low as about five hundred and twenty Years before *Christ*, and who had lived twenty-two Years in *Egypt*) was attributed the Invention of the thirty-second and forty-seventh Propositions of the first of *Euclid*; for the latter of which he conceived so much Joy, that he is said to

* Geometry, like many other Sciences, has outgrown its Name; it originally meant no more than measuring the Earth, or surveying of Land, as is plain, both from its Etymology, and the principal Use that was made of it; whereas now, it means the whole Science of Extension and Magnitude, and contemplates the Nature and Properties of all Kinds of Figures, abstractedly considered, without any Regard to Matter.

to have offer'd an Hecatomb. A Discovery of this Kind, in later Times, would have been entitled but to a small Share of Honour, and the Want of knowing these Propositions must needs make their Geometry very coarse and imperfect *. Upon this Account, therefore, it may be concluded, that the Learning of the *Egyptians*, for which their Priests were so famous, and *Moses* so celebrated in holy Writ, for having attained it, did not so much consist in Mathematics, as in the Arts of Legislation, and civil Polity, and Magic. Their Magicians or wise Men thought that the Sun, Moon, Stars, and Elements, were appointed to govern the World; and tho' they acknowledged that God might, upon extraordinary Occasions, work Miracles, reveal his Will by audible Voices, divine Appearances, Dreams or Prophecies, yet they thought, also, that generally speaking, Oracles were given, Prodigies caused, Dreams of Things to come occasioned by the Disposition of the several Parts of the Universe to influence upon one another, at the proper Places and Seasons, as constantly and as necessarily as the heavenly Bodies performed their Revolutions; and they imagined that their learned Professors, by a deep Study of, and profound Inquiry into, the Powers of Nature, could make themselves able to work Wonders, obtain Oracles and Omens, and interpret Dreams, either from Fate, (meaning the natural Course of Things) or from Nature, which was when they used any artificial Assistance by Drinks, Inebriations, Discipline, or other Means, which were thought to have a natural Power to produce the vaticinal Influence, or prophetic

 Frenzy:

* Neither was their Knowledge in Astronomy carried to any great Perfection, since they were ignorant of the true Length of the Year, taking it to contain only three hundred and sixty Days, for above a thousand Years after the Flood.

Frenzy: And in all theſe Particulars they thought the Deity not concerned, but that they were the mere natural Effects of the Influence of the Elements and Planets, at ſet Times and critical Junctures.

From *Egypt* Geometry travelled into *Greece*; for *Thales* the *Mileſian*, who flouriſhed five hundred and eighty-four Years before Chriſt, was the firſt of the *Greeks*, who, coming into *Egypt*, transferred Geometry from thence into *Greece*: He is reputed, certainly, beſides other Things, to have found out the fifth, fifteenth, and twenty-ſixth Propoſitions of *Euclid*'s firſt Book, and the ſecond, third, fourth, and fifth, of the fourth Book. The ſame Perſon improved Aſtronomy, for he began to obſerve the Equinoxes and Solſtices, and was the firſt who foretold an Eclipſe of the Sun.

After him was *Pythagoras*, of *Samos*, before mentioned: This Man much improved and adorned the mathematic Sciences, and ſo attached was he to Arithmetic in particular, that almoſt his whole Method of Philoſophizing was taken from Numbers: He firſt of all abſtracted Geometry from Matter, in which Elevation of Mind he found out ſeveral of *Euclid*'s Propoſitions. He firſt laid open the Matter of incommenſurable Magnitudes, and the five regular Bodies.

Next flouriſhed *Anaxagoras*, of *Clazomenæ*, and *Œnopides*, of *Chios*: Theſe were followed by *Briſo*, *Antipho*, and *Hippocrates*, of *Chios*; which three, for attempting the Quadrature of the Circle, were reprehended by *Ariſtotle*, and, at the ſame Time, celebrated: Then came *Democritus*, *Theodorus*, *Cyrenæus*, and *Plato*, than whom no one brought greater Luſtre to the mathematical Sciences; he amplified Geometry with great and notable Additions, beſtowing incredible Study upon it, and, above all, the Art Analytic, or of Reſolution,

was

was found out by him; the most certain Way of Invention and Reasoning. Upon the Door of his Academy was read this Inscription, οὐδεὶς ἀγεωμέτρητος εἰσίτω. Thirteen of his familiar Acquaintance are commemorated by *Proclus*, as Men by whose Studies the Mathematics were improved. After these were *Leon*, and *Eudoxus*, of *Cnidos*, a Man great in Arithmetic, and to whom we owe the whole fifth Book of the Elements; *Xenocrates*, and *Aristotle*. To *Aristeus*, *Isidore*, and *Hypsicles*, most subtile Geometricians, we are indebted for the Books of Solids. Afterwards *Euclid* gathered together the Inventions of others, disposed them into Order, improved them, and demonstrated them more accurately, and left to us those Elements, by which Youth is every where instructed in the Mathematics. He died two hundred and eighty-four Years before *Christ*. Almost an hundred Years after followed *Eratosthenes*, and *Archimedes*; the Writings of the first are lost, but we have many Remains of the latter. The very Name of *Archimedes* gives an Idea of the Top of human Subtilty, and the Perfection of the whole mathematical Sciences; his wonderful Inventions have been delivered to us by *Polybius*, *Plutarch*, *Tzetres*, and others. He was the first who was able to give the exact Quadrature or Mensuration of a Space, bounded by the Arch of a Curve and a right Line, which he did by demonstrating that the Segment of a Parabola is to its inscribed Triangle, as 4 : 3. Cotemporary with him was *Conon*; and at no great Distance of Time was *Apollonius*, of *Perga*, another Prince in Geometry, called, by Way of Encomium, The great Geometrician. We have extant four Books of Conics in his Name; tho' some think *Archimedes* was the Author of them: We have also three Books of Spherics by *Theodosius* the *Tripolite*. In the Year seventy, after *Christ*,

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appeared *Claudius Ptolomæus*, the Prince of Astronomers, a Man not only most skilful in Astronomy, but in Geometry alſo, as many other Things by him written do witneſs, but eſpecially his Books of Subtenſes. After theſe flouriſhed *Eutocius*, *Cteſibius*, *Proclus*, *Pappus*, and *Theon*. Then enſued a long Period of Ignorance; Arts and Sciences, Liberty and Learning, being driven away and over-run by that brutiſh Herd of Northern Barbarians, whoſe whole Excellence was in their Bones and Muſcles, and Feats of Chivalry their higheſt Ambition. During this diſmal Night of Ignorance, doubtleſs, many curious Diſcoveries, and uſeful Pieces of Knowledge were totally loſt, and the Remainder buried, as it were, in Ruins, till the late Reſtoration of Learning, upon the taking of *Conſtantinople* by the *Turks*, in the Year one thouſand four hundred and fifty-one, after *Chriſt*; whereby the Reſidue of *Greek* and *Roman* Learning was driven, for Refuge, into *Italy*, and the other neighbouring Countries of *Europe*.

Geometry has always heen valued for its extenſive Uſefulneſs, but has been moſt admired for its true and real Excellence, which conſiſts in its Perſpicuity and perfect Evidence: It may, therefore, be of uſe to conſider the Nature of the Demonſtrations, and the Steps by which the Ancients were able, in ſeveral Inſtances, from the Menſuration of right-lined Figures, to judge of ſuch as were bounded by curve Lines; for as they did not allow themſelves to reſolve curvilinear Figures into rectilinear Elements, it is worth While to examine, by what Art they could make a Tranſition from the one to the other.

They found that ſimilar Triangles are to each other in the duplicate Ratio of their homologous Sides; and by reſolving ſimilar Polygons into

ſimilar

similar Triangles, the same Proposition was extended to these Polygons also. But when they came to compare curvilineal Figures, which cannot be resolved into rectilineal Parts, this Method failed. Circles are the only curvilineal plain Figures considered in the Elements of Geometry. If they could have allowed themselves to have considered these as similar Polygons of an infinite Number of Sides, (as some have since done, who pretend to abridge their Demonstrations) after proving that any similar Polygons inscribed in Circles, are in the duplicate Ratio of their Diameters, they would have immediately extended this to the Circles themselves, and would have considered 2 *Euc.* 12. as an easy Corollary from the first: But there is Reason to think they would not have admitted a Demonstration of this Kind, for the old Writers were very careful to admit no precarious Principles, or ought else but a few self-evident Truths, and no Demonstrations but such as were accurately deduced from them. It was a fundamental Principle with them, that the Difference of any two unequal Quantities, by which the greater exceeds the lesser, may be added to itself till it shall exceed any proposed finite Quantity of the same Kind: And that they founded their Propositions concerning curvilineal Figures upon this Principle. in a particular Manner, is evident from the Demonstrations, and from the express Declaration of *Archimedes*, who acknowledges it to be a Foundation upon which he established his own Discourses, and cites it as assumed by the Ancients in demonstrating all the Propositions of this Kind: But this Principle seems to be inconsistent with the admitting of an infinitely little Quantity or Difference, which, added to itself any Number of Times, is never supposed to become equal to any finite Quantity soever.

They proceeded, therefore, in another Manner, less direct indeed, but perfectly evident. They found that the inscribed similar Polygons, by having the Number of their Sides increased, continually approached to the Areas of the Circles; so that the decreasing Difference between each Circle and its inscribed Polygon, by still further and further Divisions of the circular Arches, which the Sides of the Polygon subtend, could become less than any Quantity that could be assigned; and that all this while the similar Polygons observed the same constant invariable Proportions to each other, *viz.* that of the Squares of the Diameters of the Circles. Upon this they founded a Demonstration, that the Proportion of the Circles themselves could be no other than that same invariable Ratio of the similar inscribed Polygons. For they proved, by the Doctrine of Proportions only, that the Ratio of the two inscribed Polygons cannot be the same as the Ratio of one of the Circles to a Magnitude less than the other, nor the same as the Ratio of one of the Circles to a Magnitude greater than the other; therefore the Ratio of the Circles to each other, must be the same as the invariable Ratio of the similar Polygons inscribed in them, which is the Duplicate of the Ratio of the Diameters.

In the same Manner the Ancients have demonstrated, that Pyramids of the same Height are to each other as their Bases, that Spheres are as the Cubes of their Diameters, and that a Cone is the one third Part of a Cylinder on the same Base, and of the same Height. In general, it appears from their Way of Demonstration, that when two variable Quantities, which always have an invariable Ratio to each other, approach at the same Time to two determined Quantities, so that they may differ less from them than by any assignable

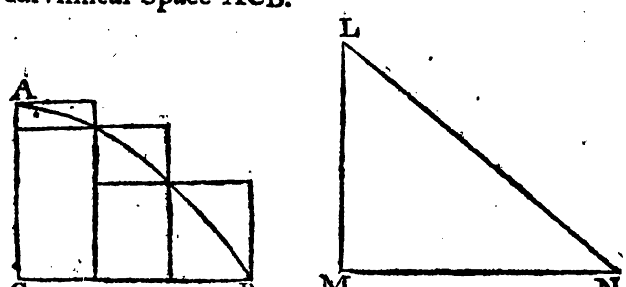
Measure;

Measure; the Ratio of these Limits or determined Quantities must be the same as the invariable Ratio of the two variable Quantities: And this may be considered as the most simple and fundamental Proposition in this Doctrine, by which we are enabled to compare curvilineal Spaces in some of the more simple Cases.

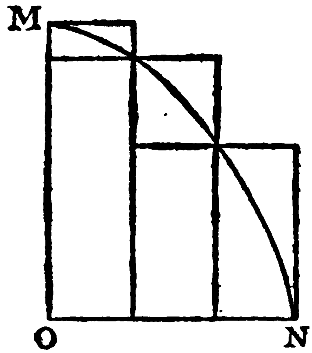
The next Improvement in the Way of demonstrating among the ancient Geometricians, seems to be that which we call the Method of Exhaustions, which, for the further Illustration of this Subject, may be represented thus.

Suppose there are two curvilineal Spaces, ACB and MON; wherein are drawn Parallelograms, whose Breadth may be continually diminished; it is then obvious, that the first circumscribing, and last inscribing Figures, may be made to differ from that curvilineal Space ACB, and from each other, by less than any Space, how minute soever, that shall be named; *i. e.* the circumscribed Figure can be made less than any Space that exceeds the Curve, and the inscribed Figure greater than any Space that is less than the Curve.

If by considering the Properties of these inscribed and circumscribed Figures, which arise from the Nature of the Curve they are adapted to, a right-lined Space LMN may be assigned, that shall be greater than every inscribed Figure, and less than every circumscribed Figure, this right-lined Space LMN may be proved to be equal to the curvilineal Space ACB.



For were it greater, a circumſcribed Figure might be made leſs; and if it were leſs, an inſcribed Figure might be made greater.



If, therefore, Parallelograms, whoſe Breadth may be any how diminiſhed, are drawn inſcribing and circumſcribing theſe Curves; and if they are deſcribed in ſuch a Manner, that the circumſcribed Figure of one Curve to the circumſcribed Figure of the other, and the inſcribed to the inſcribed, has one and the ſame conſtant Proportion in every Deſcription: I ſay, that the curve Figure ACB, is to the Curve MON, in the ſame Proportion which the inſcribed and circumſcribed Figures conſtantly bear to each other.

For no Space greater than ACB can have to MON this Ratio, ſince if it could, a Figure might be circumſcribed about ACB leſs than this ſuppoſed greater Space; and this circumſcribed Figure, to the correſponding Figure circumſcribing MON, would be in the ſame Ratio as the ſuppoſed greater Space to the Curve MON; *i. e.* four Quantities being in the ſame Proportion, the firſt would be leſs than the third, and the ſecond greater than the fourth. Nor can any Space leſs than ACB have to MON the conſtant Ratio of the Figures in one Curve to the Figures in the

other. For if it could, a Figure might be inſcribed within ACB, which would be greater than this ſuppoſed leſſer Space; and this inſcribed Figure, to its correſpondent Figure inſcribing MON, would be in the ſame Ratio as this imagined leſſer Space to the Curve MON; *i. e.* four Quantities being in the ſame Proportion, the firſt would be greater than the third, and the ſecond leſs than the fourth. Thus no Space but ACB can be to MON in the conſtant Ratio of the circumſcribed and inſcribed Figures.

In the Manner here deſcribed did the antient Geometricians demonſtrate whatever they diſcovered relating to the Dimenſions or Proportions of curve Lines, curvilineal Spaces, and Solids bounded by curve Surfaces; and of which, Sir *Iſaac Newton*'s Doctrine of prime and ultimate Ratios, is no other than an Abbreviation or Improvement in the Form.

Archimedes, indeed, takes a different Way for comparing the Spheroid with the Cone and Cylinder, that is more general, and has a nearer Analogy to the modern Methods. He ſuppoſes the Terms of a Progreſſion to increaſe conſtantly by the ſame Difference, and demonſtrates ſeveral Properties of ſuch a Progreſſion relating to the Sum of the Terms, and the Sum of their Squares; by which he is able to compare the parabolic Conoid, the Spheroid, and hyperbolic Conoid, with the Cone; and the Area of his ſpiral Line with the Area of the Circle. There is an Analogy betwixt what he has ſhewn of theſe Progreſſions, and the Proportions of Figures demonſtrated in the Elementary Geometry; the Conſequence of which may illuſtrate his Doctrine, and ſerve, perhaps, to ſhew that it is more regular and compleat in its Kind than ſome have imagined. The Relation of the Sum of the Terms to the Quantity that ariſes by taking

the greateſt of them as often as there are Terms, is illuſtrated by comparing the Triangle with a Parallelogram of the ſame Height and Baſe; and what he has demonſtrated of the Sum of the Squares of the Terms compared with the Square of the greateſt Term, may be illuſtrated by the Proportion of the Pyramid to the Priſm, or of the Cone to the Cylinder, their Baſes and Heights being equal; and by the Ratios of certain Fruſtums or Proportions of theſe Solids deduced from the elementary Proportions.

He appears ſolicitous, that his Demonſtrations ſhould be found to depend on thoſe Principles only, that had been univerſally received before his Time. In his Treatiſe of the Quadrature of the Parabola, he mentions a Progreſſion, whoſe Terms decreaſe conſtantly in the Proportion of four to one; but he does not ſuppoſe this Progreſſion to be continued to Infinity, or mention the Sum of an infinite Number of Terms; tho' it is manifeſt, that all which can be underſtood by thoſe who aſſign that Sum, was fully known to him. He appears to have been more fond of preſerving to the Science all its Accuracy and Evidence, than of advancing Paradoxes; and contents himſelf with demonſtrating this plain Property of ſuch a Progreſſion, That the Sum of the Terms continued at Pleaſure, added to the third Part of the laſt Term, amounts always to $\frac{4}{3}$ of the firſt Term: Nor does he ſuppoſe the Chords of the Curve to be biſected to Infinity; ſo that after an infinite Biſection, the inſcribed Polygon might be ſaid to coincide with the Parabola. Theſe Suppoſitions had been new to the Geometricians in his Time, and ſuch he appears to have carefully avoided.

This is a ſummary Account of the Progreſs that was made by the Ancients in meaſuring and comparing curvilineal Figures, and of the Method by which

which they demonſtrated their Theorems of this Kind. It is often ſaid, that curve Lines have been conſidered by them as Polygons of an infinite Number of Sides; but this Principle no where appears in their Writings: We never find them reſolving any Figure or Solid into infinitely ſmall Elements: On the contrary, they ſeem to have avoided ſuch Suppoſitions, as if they judged them unfit to be received into Geometry, when it was obvious, that their Demonſtrations might have been ſometimes abridged by admitting them. They conſidered curvilineal Areas as the Limits of circumſcribed or inſcribed Figures, of a more ſimple Kind, which approach to theſe Limits (by a Biſection of Lines or Angles, continued at Pleaſure) ſo that the Difference between them may become leſs than any given Quantity. The incribed or circumſcribed Figures were always conceived to be of a Magnitude and Number that is aſſignable; and from what had been ſhewn of theſe Figures, they demonſtrated the Menſuration, or the Proportions of the curvilineal Limits themſelves, by Arguments *ab Abſurdo*. They had made frequent Uſe of Demonſtrations of this Kind from the Beginning of the Elements; and theſe are, in a particular Manner, adapted for making a Tranſition from right-lined Figures, to ſuch as are bounded by curve Lines. By admitting them only, they eſtabliſhed the more difficult and ſublime Part of their Geometry, on the ſame Foundation as the firſt Elements of the Science; nor could they have propoſed to themſelves a more perfect Model.

But as theſe Demonſtrations, by determining diſtinctly all the ſeveral Magnitudes and Proportions of theſe inſcribed and circumſcribed Figures, did frequently extend to very great Lengths, other Methods of demonſtrating have been contrived

trived by the Moderns, whereby to avoid these circumstantial Deductions. The first Attempt of this Kind known to us, was made by *Lucas Valerius*; but afterwards *Cavalerius*, an *Italian*, about the Year one thousand six hundred and thirty five, advanced his Method of Indivisibles, in which he proposes, not only to abbreviate the antient Demonstrations, but to remove the indirect Form of Reasoning used by them, of proving the Equality or Proportion between Lines and Spaces, from the Impossibility of their having any different Relation; and to apply to these curved Magnitudes the same direct Kind of Proof that was before applied to right-lined Quantities.

This Method of comparing Magnitudes, invented by *Cavalerius*, supposes Lines to be compounded of Points, Surfaces of Lines, and Solids of Planes; or, to make use of his own Description, Surfaces are considered as Cloth, consisting of parallel Threads; and Solids are considered as formed of parallel Planes, as a Book is composed of its Leaves, with this Restriction, that the Threads or Lines, of which Surfaces are compounded, are not to be of any conceivable Breadth, nor the Leaves or Planes of Solids of any Thickness. He then forms these Propositions, that Surfaces are to each other, as all the Lines in one to all the Lines in the other; and Solids, in like Manner, in the Proportion of all the Planes.

This Method exceedingly shortened the former tedious Demonstrations, and was easily perceived; so that Problems, which at first Sight appeared of an insuperable Difficulty, were afterwards resolved, and came, at length, to be despised, as too simple and easy: The Mensuration of Parabolas, Hyperbolas, Spirals of all the higher Orders, and the famous Cycloid, were among the early Productions of this Period. The Discoveries

ries made by *Torricelli*, *de Fermat*, *de Roberval*, *Gregory*, *St. Vincent*, &c. are well known. They who have not read many Authors, may find a Synopsis of this Method in *Ward's Young Mathematician's Guide*, where he treats of the Mensuration of Superficies and Solids.

Notwithstanding, as this Method is here explained, it is manifestly founded on inconsistent and impossible Suppositions; for while the Lines, of which Surfaces are supposed to be made up, are real Lines of no Breadth, it is obvious, that no Number, whatever, of them, can form the least imaginable Surface: If they are supposed to be of some sensible Breadth, in order to be capable of filling up Spaces, *i. e.* in Reality to be Parallelograms, how minute soever be their Altitude, the Surfaces may not be to each other in the Proportion of all such Lines in one, to all the like Lines in the other; for Surfaces are not always in the same Proportion to each other with the Parallelograms inscribing them.

The same contradictory Suppositions do obviously attend the Composition of Solids by parallel Planes, or of Lines by such imaginary Points.

This heterogeneous Composition of Quantity, and Confusion of its Species, so different from that Distinctness, for which the Mathematics were ever famous, was opposed at its first Appearance by several eminent Geometricians, particularly by *Guldinus* and *Tacquet*, who not only excepted to the first Principles of this Method, but taxed the Conclusions formed upon it as erroneous. But as *Cavalerius* took Care, that the Threads or Lines of which the Surfaces to be compared together were formed, should have the same Breadth in each (as he himself expresses it) the Conclusions deduced by his Method, might generally be verified by sounder Geometry; since the Comparison of these Lines

Lines was, in Effect, the comparing together the inſcribed Figures.

As in the Application of this Method, Error, by proper Caution, might be avoided, the Aſſiſtance it ſeemed to promiſe in the analytical Part of Geometry, made it eagerly followed by thoſe who were more deſirous to diſcover new Propoſitions, than ſolicitous about the Elegance or Propriety of their Demonſtrations. Yet ſo ſtrange did the contradictory Conception appear, of compoſing Surfaces out of Lines, and Solids out of Planes, that, in a ſhort Time, it was new modelled into that Form, which it ſtill retains, and which now univerſally prevails among the foreign Mathematicians, under the Name of the differential Method, or the Analyſis of infinitely Littles.

In this reformed Notion of Indiviſibles, Surfaces are now ſuppoſed as compoſed not of Lines, but of Parallelograms, having infinitely little Breadths and Solids, in like Manner as found of Priſms, having infinitely little Altitudes. By this Alteration it was imagined, that the heterogeneous Compoſition of *Cavalerius* was ſufficiently evaded, and all the Advantages of his Method retained. But here, again, the ſame Abſurdity occurs as before; for if by the infinitely little Breadth of theſe Parellelograms, we are to underſtand what theſe Words literally import, *i. e.* no Breadth at all; then they cannot, any more than the Lines of *Cavalerius*, compoſe a Surface; and if they have any Breadth, the right Lines bounding them cannot coincide with a Surface bounded by a curve Line.

The Followers of this new Method grew bolder than the Diſciples of *Cavalerius*, and having tranſformed his Points, Lines, and Planes, into infinitely little Lines, Surfaces, and Solids, they pretended, they no longer compared together heterogeneous

rogeneous Quantities, and insisted on their Principles, being now become genuine; but the Mistakes they frequently fell into, were a sufficient Confutation of their Boasts; for, notwithstanding this new Model, the same Limitations and Cautions were still necessary: For Instance, this Agreement between the inscribing Figures and the curved Spaces to which they are adapted, is only partial; and in applying their Principles to Propositions already determined by a juster Method of Reasoning, they easily perceived this Defect; both in Surfaces and Solids, it was evident, at first View, that the Perimeters disagreed. And as no one Instance can be given, where these indivisible or infinitely little Parts do so compleatly coincide with the Quantities they are supposed to compound, as in every Circumstance to be taken for them, without producing erroneous Conclusions, so we find, where a surer Guide was wanting, or disregarded, these Figures were often imagined to agree, where they ought to have been supposed to differ.

Leibnitz, in two Dissertations, one on the Resistance of Fluids, the other on the Motion of the heavenly Bodies, has, on this Principle, reasoned falsly concerning the Lines intercepted between Curves and their Tangents. *Berneulli* has, likewise, made the same Mistake in a Dissertation on the Resistance of Fluids, and in a pretended Solution of the Problem concerning isoperimetrical Curves. Nay, Mr. *Parent* has had the Rashness to oppose erroneous Deductions from this absurd Principle, to the most indubitable Demonstrations of the great *Huygens*. Thus it appears, that the Doctrine of Indivisibles contains an erroneous Method of Reasoning, and, in Consequence thereof, in every new Subject to which it shall be applied, is liable to fresh Errors.

It is alſo manifeſt, that the great Brevity it gave to Demonſtrations, aroſe entirely from the abſurd Attempt of comparing curvilineal Spaces in the ſame direct Manner as right-line Figures can be compared; for, in order to conclude directly the Equality or Proportion of ſuch Spaces, no Scruple was made of ſuppoſing, contrary to Truth, that rectilineal Figures, capable of ſuch direct Compariſon, could adequately fill up the Spaces in Queſtion; whereas, the Doctrine of Exhauſtions does not attempt, from the Equality or Proportion of the inſcribing or circumſcribing Figures, to conclude, directly, the like Proportions of theſe Spaces, becauſe thoſe Figures can never, in Reality, be made equal to the Spaces they are adapted to: But as theſe Figures may be made to differ from the Spaces to which they are adapted, by leſs than any Space propoſed, how minute ſoever, it ſhews, by a juſt, tho' indirect Deduction from theſe circumſcribing and inſcribing Figures, that the Spaces whoſe Equality is to be proved, can have no Difference; and that the Spaces, whoſe Proportion is to be ſhewn, cannot have a different Proportion than that aſſigned them.

The *Arithmetica Infinitorum* of Dr. *Wallis*, was the fulleſt Treatiſe of this Kind that appeared before the Invention of Fluxions. *Archimedes* had conſidered the Sums of the Terms in an arithmetical Progreſſion, and of their Squares only, (or rather the Limits of theſe Sums only) theſe being ſufficient for the Menſuration of the Figures he had examined. Dr. *Wallis* treats this Subject in a very general Manner, and aſſigns like Limits for the Sums of any Powers of the Terms, whether the Exponents be Integers or Fractions, poſitive or negative. Having diſcovered one general Theorem that includes all of this Kind, he then com-

poſed

posed new Progressions from various Aggregates of these Terms, and enquired into the Sums of the Powers of these Terms, by which he was enabled to measure accurately, or by Approximation, the Areas of Figures without Number: But he composed this Treatise (as he tells us) before he had examined the Writings of *Archimedes*; and he proposes his Theorems and Demonstrations in a less accurate Form: He supposes the Progressions to be continued to Infinity, and investigates, by a Kind of Induction, the Proportion of the Sum of the Powers, to the Production that would arise by taking the greatest Power as often as there are Terms. His Demonstrations, and some of his Expressions, (as when he speaks of Quantities more than infinite) have been excepted against; however, it must be owned, this valuable Treatise contributed to produce the great Improvements which soon after followed.

But Sir *Isaac Newton* has accomplished what *Cavalerius* wished for, by inventing the Method of Fluxions, and proposing it in a Way that admits of strict Demonstration, which requires the Supposition of no Quantities, but such as are finite, and easily conceived; by his Doctrine of prime and ultimate Ratios, he has found out the proper Medium, whereby to avoid the impossible Notion of Indivisibles on the one hand, and the Length of Exhaustions on the other. The Computations in this Method, are the same as in the Method of Infinitesimals, but it is founded on accurate Principles, agreeable to the antient Geometry; in it the Premises and Conclusions are equally accurate, no Quantities are rejected as infinitely small, and no Part of a Curve is supposed to coincide with a right Line: But as the Explication of their Nature and Use has

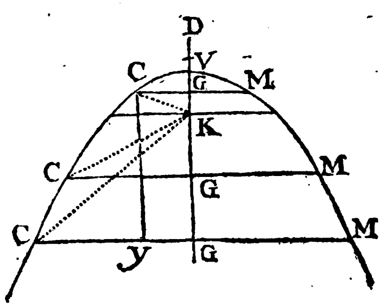
employed some of our greatest Mathematicians to write express Treatises thereon; and as the Invention can never be sufficiently applauded, we will conclude with Mr. *Ditton*, that the next Improvement must be the Science of pure Intelligences.

CONIC SECTIONS.

Of the PARABOLA.

The GENESIS.

IF from a Point V, in any indefinite right Line, there be taken VD=VK, and from the Point K, as a Center, with the Distance DG, you intersect CM drawn perpendicular to DG, those Points will be in the Curve of a Parabola; and proceeding in this Manner, an indefinite Number of Points may be found, thro' which, if a Line be supposed drawn, the Space (CVM) comprehended thereby, and any right Line drawn at right Angles, to the above indefinite Line, will be a Parabola.



DEFI-

Definitions.

1. The Point V is the Vertex, and K the Focus of the Parabola.

2. The right Line DG, paſſing thro' the Focus, is called the Axis.

3. A right Line drawn perpendicular to the Axis, terminating in the Curve, is called an Ordinate to the Axis, as GC.

4. The Diſtance, in the Axis, from the Vertex to the Interſection of the Ordinate, is called the Abſciſſa, correſponding to that Ordinate, as VG.

5. A right Line drawn from any Point in the Curve, and parallel to the Axis, is called a Diameter, as CY; and the Point in the Curve, from whence it is drawn, the Vertex of that Diameter.

Proposition I.

The Square of any Ordinate, is equal to the Rectangle of the Abſciſſa of that Ordinate, drawn into quadruple the Diſtance of the Focus from the Vertex; that is, $\overline{GC}^2 = VG \times 4\,KV$.

Demonstration.

Let $KV = VD = q$, $VG = x$, and $GC = y$; then, by the Geneſis, $GK = q \backsim x$, and $DG = CK = q + x$; but (by *Eu.* 1. 47.) $\overline{KC}^2 = \overline{KG}^2 + \overline{GC}^2$; that is, $q^2 + 2qx + x^2 = q^2 - 2qx + x^2 + y^2$, or $4qx = y^2$; that is, $\overline{GC}^2 = VG \times 4\,KV$. Q.E.D.

Corollary I.

The Squares of the Ordinates, are to each other, as their reſpective Abſciſſas; becauſe $Y^2 = 4qX$, and $y^2 = 4qx$; therefore, $Y^2 : y^2 :: 4qX : 4qx :: X : x$.

Defi-

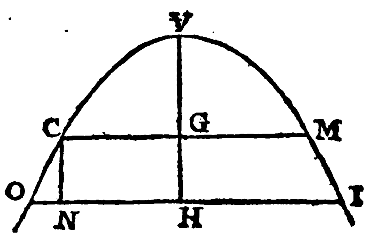
Definition. Quadruple the Diſtance of the Focus from the Vertex, is called the Parameter of the Axis, and is a third Proportional to any Abſciſſa, and its correſponding Ordinate. For by putting $4q=p$, we have $px=y^2$; therefore, $x:y::y:p$.

Corollary II.

The Ordinate paſſing thro' the Focus, is equal to Half the Parameter of the Axis. For in this Caſe $x=q$; therefore (by the Propoſition) $4q^2=y^2$; whence $y=2q=\frac{1}{2}p$.

Proposition II.

As the Parameter of the Axis, is to the Sum of any two Ordinates, ſo is their Difference, to the Difference of their Abſciſſas; that is, $p:IN::NO:NC$.



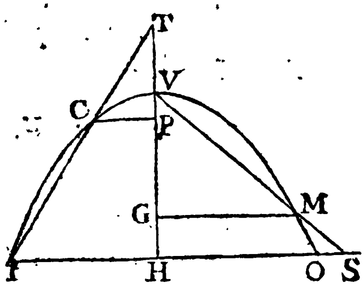
Demonstration.

Let $HO=Y$, $GC=y$, $VH=X$, and $VG=x$; then (by Prop. 1.) $pX=Y^2$, and $px=y^2$; therefore, $pX-px=Y^2-y^2$; conſequently, $p:Y+y::Y-y:X-x$; that is, $p:IN::NO:NC$. Q. E. D.

PRO-

PROPOSITION III.

If from the Vertex a right Line be drawn, so as to cut the Curve, and continued till it meet any Ordinate produced, it will be, as the Parameter of the Axis, is to the Ordinate drawn from the Intersection with the Curve, so is the produced Ordinate, to its Abscissa; that is, p : GM :: HS : HV.



DEMONSTRATION.

Let HS $= b$, VG $= x$, VH $=$ X, HO $=$ Y, and GM $= y$; then (by Prop. 1.) $y^2 : Y^2 :: x : X ::$ by similar Triangles, $y : b$; therefore, $\frac{by}{y} = Y^2 = pX$; or $by = pX$; that is, p : GM :: HS : HV. Q.E.D.

PROPOSITION IV.

If from any Point D, in the Axe produced, a right Line be drawn intersecting the Curve in two Points C and I, and the Ordinates CP, IH, be drawn from the said Intersections; VD will be a mean Proportional between VP and VH.

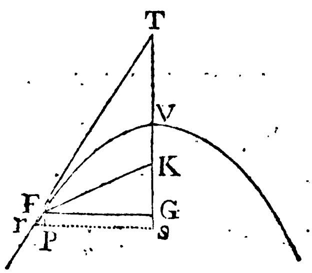
DEMON-

DEMONSTRATION.

Let $VD=b$, $VP=x$, $VH=X$; then $PD=b+x$, and $HD=b+X$; but from similar Triangles $\overline{PD}^2 : \overline{HD}^2 :: \overline{PC}^2 : \overline{TH}^2 ::$ (by Prop. 1.) $PV : HV$; that is, $b^2+2bx+x^2 : b^2+2bX+X^2 :: x : X$; therefore $\overline{X-x}\times b^2 = \overline{X-x}\times xX$; or $b^2=xX$; that is, $x:b::b:X$; or $VP : VD :: VD : HV$. Q. E. D.

PROPOSITION V.

If any right Line touch the Curve, and an Ordinate be drawn from the Point of Contact; then the Abscissa corresponding that Ordinate, will be equal to the Distance, in the Axe produced, from the Vertex of the Curve, to the Intersection of the Tangent; that is, $GV=VT$.



DEMONSTRATION.

Let rF be an indefinitely small Part of the Curve, and continued to T; draw rs parallel to the Ordinate, and Fp parallel to the Axe, and let $Fp=n$, $rp=m$, $VT=a$, and the other Sym-

bols remaining as usual; then $VS = n + x$, and $r,s = m + y$; whence, by similar Triangles, $m : n :: y : a + x$; therefore $\frac{ny}{m} = a + x$; but (by Prop. 1.) $p \times VS = \overline{Sr}^2$, and $p \times VG = \overline{GF}^2$; that is, $pn + px = m^2 + 2my + y^2$, and $px = y^2$; therefore $y^2 + 2my - pn = px = y^2$; that is, $n = \frac{2my}{p}$, and consequently $x + a = \frac{2my}{p} \times \frac{y}{m} = \frac{2y^2}{p} = \frac{2px}{p} = 2x$; therefore $a = x$, or $VT = GV$. Q. E. D.

Proposition VI.

If from the Point of Contact, a right Line be drawn to the Focus, it will be equal to the Distance, in the Axe produced, from the Focus to the Intersection of the Tangent; that is, $KF = KT$.

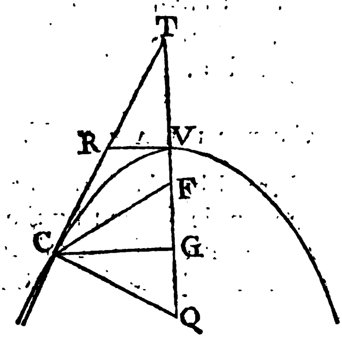
Demonstration.

By the last (Proposition) $GT = 2x$, and by the first, $KG = x - \frac{1}{4}p$; consequently $KT = GT - KG = 2x - x + \frac{1}{4}p = x + \frac{1}{4}p$. = by the Genesis to KF. Q. E. D.

Proposition VII.

If to the Tangent, from the Point of the Contact, a Perpendicular be drawn, and produced to meet the Axe; then the Distance in the Axe from that Point, to the Ordinate drawn from the Point

of Contact, that is, the *Subnormal*, is equal to half the Parameter of the Axe; that is, $QG = \frac{1}{2}p$.



DEMONSTRATION.

Let $QG = b$; then (by *Eu.* 8. 6.) $GT : GC :: GC : GQ$; that is, $2x : y :: y : b$; therefore, $2bx = y^2 = px$; whence $b = \frac{1}{2}p$, or $QG = \frac{1}{2}p$. Q.E.D.

PROPOSITION VIII.

The Diſtance from the Focus to the Point of Contact; from the Focus to the Interſection of the Tangent with the Axe, and from the Focus to the End of the Subnormal, are equal to each other; that is, $FC = FT = FQ$.

DEMONSTRATION.

From the Geneſis $GF = x - \frac{1}{4}p$, and (by Prop. 7th) $QG = \frac{1}{2}p$; therefore $FG = GF + GQ = x + \frac{1}{4}p =$ (by the Geneſis) $FC =$ (by Prop. 6th) FT. Q.E.D.

 COROL.

COROLLARY I.

Hence F is the Center of a Circle, the Periphery of which will paſs through the Points denoted by Q, C, and T.

COROLLARY II.

The Angle formed by the Tangent and Axe, is equal to half the Angle formed by the Axe, and a right Line drawn from the Point of Contact; that is, the Angle CTG $= \frac{1}{2}$ the Angle CFG, by *Eu.* 31. 3.

PROPOSITION IX.

If from the Vertex, a right Line be drawn parallel to an Ordinate, drawn from the Point of Contact, to meet the Tangent; the Square of that Line, will be equal to the Rectangle of half the Parameter of the Axis, drawn into half the Abſciſſa correſponding that Ordinate; that is, $\overline{VR}^2 = \frac{1}{2}p \times \frac{1}{2}$ GV.

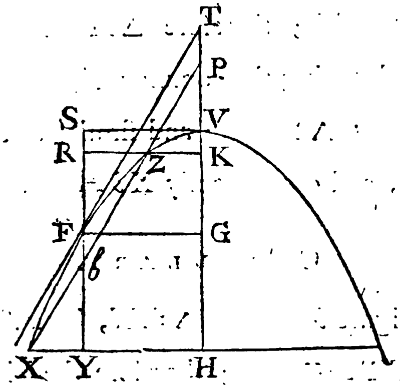
DEMONSTRATION.

Let VR $= b$; then, becauſe the Triangles TVR, TGC are ſimilar, and VT $= \frac{1}{2}$ GT (by Prop. 5.) therefore, VR $= \frac{1}{2}$ GC; that is, $b = \frac{1}{2}y$; whence $b^2 = \frac{1}{4}y^2 =$ (by Prop. 1.) $\frac{1}{2}p \times \frac{1}{2}x$, or $\overline{VR}^2 = \frac{1}{2}p \times \frac{1}{2}$ GV. Q. E. D.

PROPOSITION X.

If to the Tangent drawn to the Vertex of any Diameter, a right Line be drawn parallel; the Part of

of that Line which lies within the Curve, will be bissected by that Diameter; that is, the Ordinate $Xb = bZ$.



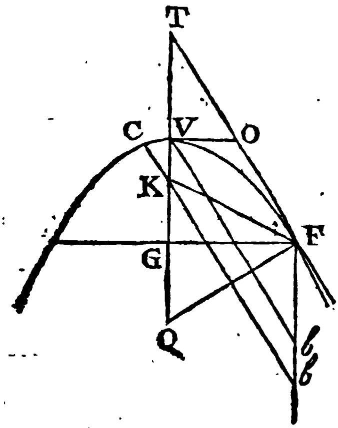
DEMONSTRATION.

Draw KR, VS parallel to the Ordinate FG, meeting the Diameter FY produced in R and S; through F, where that Diameter intersects the Curve, draw the Tangent FT, and through Z, where KR intersects the Curve, draw PX parallel to that Tangent, meeting the Curve in X; then,

1. Because GT is double VG (by Prop. 5.) the Triangle $FGT = \square SG$, and since PX is parallel to TF, the Triangles KZP, FGT are similar; wherefore $\triangle GFT$, or $\square GS : \triangle KZP :: \overline{GF}^2 : \overline{KZ}^2 ::$ (by Cor. 1. Prop. 1.) $GV : KV ::$ (by *Eu.* 1. 6.) $\square GS : \square KS$; therefore $\triangle KZP = \square KS$.

2. The

Abſciſſa of the Ordinate, drawn from the Vertex of that Diameter; that is, $P = p + 4GV$.



From the Vertex draw Vp parallel to the Tangent FT, which (by Prop. 10.) will be an Ordinate to the Diameter FY; then by reaſon of Parallels $bF = VT =$ (by Prop. 5.) $GV = x$; and by the laſt, $Px = \overline{bV}^2 = \overline{FT}^2 = \overline{FG}^2 + \overline{GT}^2$ $= 4x^2 + y^2 = 4x^2 + px$; conſequently $P = 4x + p = p + 4GV$. Q.E.D.

PROPOSITION XIII.

The Diſtance from the Focus, to the Vertex of any Diameter, is equal to one Fourth of the Parameter of that Diameter; that is, $KF = \frac{1}{4}P$.

DEMONSTRATION.

By the laſt Propoſition, $P = p + 4VT$, and by the firſt, $p = 4KV$; therefore $P = 4KV + 4VT$,

and

and consequently $\frac{1}{4}P = KV + VT = KT =$ (by Prop. 6.) KF. Q. E. D.

PROPOSITION XIV.

If from the Focus, a Perpendicular be drawn to any Tangent; then the Square of that Line, will be equal to the Rectangle under the Focal-distance, and the Distance of the Point of Contact from the Focus; that is, $\overline{KO}^2 = KV \times KF$.

DEMONSTRATION.

From the Vertex V, draw VO parallel to GF, which will coincide with the Point O; because (by Prop. 5.) $GV = VT$; therefore (by *Eu.* 2. 6.) $TO = OF$; also because the Angle KOT is right (by *Eu.* 8. 6.) $TK : KO :: KO : KV$; consequently $\overline{KO}^2 = KV \times TK = KV \times KF$. Q. E. D.

PROPOSITION XV.

If an Ordinate to any Diameter pass thro' the Focus; then the Abscissa corresponding to that Ordinate, will be equal to one Fourth, and the Ordinate equal to one Half of the Parameter corresponding to that Diameter.

DEMONSTRATION.

Because bC is parallel to FT, and bF parallel to KT; therefore $bF = KT =$ (by Prop. 6.) $KF =$ (by Prop. 13.) $\frac{1}{4}P$.

 2. Since

2. Since $bF = \frac{1}{4}P$, and (by Prop. 11.) $P \times bF = \overline{bC}^2$; therefore $\frac{1}{4}P^2 = \overline{bC}^2$, and consequently $\frac{1}{2}P = bC$. Q. E. D.

Proposition XVI.

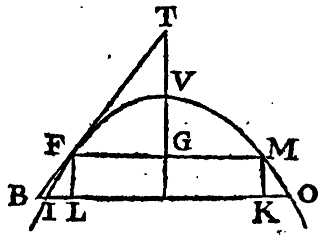
The Diſtance in the Axis, from the Interſection of the Tangent, to the End of the Subnormal, is equal to half the Parameter of that Diameter, whoſe Vertex is the Point of Contact; that is $QT = \frac{1}{2}P$.

Demonstration.

Since (by Prop. 13) $FK = \frac{1}{4}P$, and (by Prop. 8.) $FK = QK = KT$; conſequently $QT = QK + KT = \frac{1}{4}P + \frac{1}{4}P = \frac{1}{2}P$. Q. E. D.

Proposition XVII.

If a double Ordinate be drawn from the Point of Contact, and another double Ordinate be drawn below to meet the Tangent produced; then as the double Ordinate paſſing through the Point of Contact, is to the Sum of the two Ordinates, ſo is their Difference, to the external Part of the lower Ordinate added to the Difference of the Ordinates; that is, $MF : OL :: IL : BL$.



DEMON-

DEMONSTRATION.

Let $VG = x$, then (by Prop. 5.) $GT = 2x$, $FG = y$, $OL = c$, $IL = m$, and $LB = d$; then (by Prop. 2.) $p : c :: m : \frac{cm}{p} = LF$, and by similar Triangles, $2x : y :: \frac{mc}{p} : d$; therefore $2pdx = mcy$; but because $px = y^2$; (by Prop. 1.) $2dy = mc$, or $2y : c :: m : d$; that is, $MF : OL :: IL : BL$. Q. E. D.

PROPOSITION XVIII.

The same Things being supposed as before; the Difference of the Ordinates, is a Mean-proportional, between the Double of the upper Ordinate, and the external Part of the lower; that is, $FM : IL :: IL : BI$.

DEMONSTRATION.

Let $BI = c$, $IL = m$, and $FM = 2y$; then $OL = m + 2y$, and (by Prop. 17.) $2y : m + 2y :: m : d$; therefore $d = \frac{m^2 + 2my}{2y}$, and consequently $c = d - m = \frac{m^2 + 2my}{2y} - m = \frac{m^2}{2y}$; that is, $FM : IL :: IL : BL$. Q. E. D.

PROPOSITION XIX.

The same Things being still supposed; as the Double of the lower Ordinate added to the external Part, is to the Sum of the two Ordinates, so is the external Part of the lower Ordinate added to the Difference of the Ordinates, to the Difference of the Ordinates; that is, OB : LB :: OL : IL.

DEMONSTRATION.

Let OL $= c$, LB $= d$, IL $= m$; then OB $= c + d$, and MF $= c - m$; but (by Prop. 17.) MF : OL :: IL : BL; that is, $c - m : c :: m : d$; therefore $cd - dm = cm$, and consequently $cm + dm = cd$; or $c + d : c :: d : m$; that is, OB : LB :: OL : IL. Q. E. D.

PROPOSITION XX.

Still supposing the same Things; having OI, and BI given, it is proposed to find IL.

SOLUTION.

Let KL $= b$, IL $= m$, BI $= a$, and OI $= c$; then (by Prop. 18) KL : (MF) IL :: IL : BI; that is, $b : m :: m : a$; therefore $ba = m^2$ and $b = \frac{m^2}{a}$; also $c = b + 2m = \frac{m^2}{a} + 2m = \frac{m^2 + 2am}{a}$; there-

fore

fore $ac = m^2 + 2am$, and, consequently, by Reduction, $m = \sqrt{\overline{a+c} \times a} - a$, Q. E. I.

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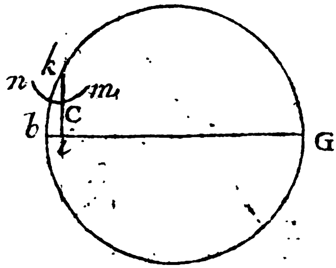
fore $ac = m^2 + 2am$, and, consequently, by Reduction, $m = \sqrt{\overline{a+c} \times a} - a$, Q.E.I.



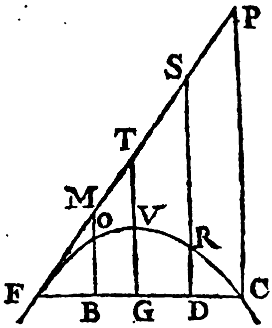
COROLLARY.

Hence from a Point B, without the Curve, and not in the Axe produced, may be drawn a Tangent. For if from the given Point B, the right Line BO be drawn perpendicular to the Axe, meeting the Curve in O; let bG be taken equal to $2BI + OI$, about which as a Diameter, let a Circle be described; in that Diameter, take bi equal to BI, and from the Point i, erect the Perpendicular ik, meeting the Periphery in k; from k as a Center, with the Interval BI, describe ncm cutting ik in C; lastly, in IO, take IL equal to ci, and draw LF parallel to the Axe, and the Point F will be determined; through which if a right Line be drawn from the given Point B, terminating in the Axe produced, it will be a Tangent to the Curve.

PROPOSITION XXI.

If FP touch the Curve in F, and from any two Points M, S in that Tangent, the right Lines BM, SD be drawn parallel to the Axe, meeting the Ordinate

dinate in B and D; then will $MO:SR :: \overline{FB}^2 : \overline{FD}^2$.



DEMONSTRATION.

Let $MO = b$, $FB = c$, $SR = d$, and $FD = q$, alſo $GV = VT = x$; then (by Prop. 11.) $x:b :: \overline{FT}^2 : \overline{FM}^2 ::$ (by ſimilar Triangles) $y^2:c^2$; and $x:d :: \overline{FT}^2 : \overline{FS}^2 ::$ (by ſimilar Triangles) $y^2 : q^2$; therefore, by Equality, $b:d :: c^2 : q^2$, or $MO : SR :: \overline{FB}^2 : \overline{FD}^2$. Q.E.D.

PROPOSITION XXII.

If from any Point in the Tangent, a right Line be drawn parallel to the Axe, meeting an Ordinate; the Rectangle of the Parameter of the Axe, into the external Part of that Line, will be equal to the Square of the Segment of the Ordinate, intercepted between that Line and the Point of Contact; that is, $p \times MO = \overline{FB}^2$, or $p \times RS = \overline{FD}^2$.

DEMON-

DEMONSTRATION.

By the laſt Propoſition, $\frac{c^2x}{b}=y^2=$ (by Prop. 1.) px; alſo $\frac{q^2x}{d}=y^2=px$; therefore $pb=c^2$, and $dp=q^2$; that is, $p\times MO=\overline{FB}^2$, and $p\times RS=\overline{FD}^2$. Q.E.D.

PROPOSITION XXIII.

If FP touch the Parabola in F, and if from any Point S, in the Tangent, a right Line SD be drawn parallel to the Axe, cutting another right Line FC, drawn from the Point of Contact any how within the Curve; then the Curve will cut the firſt Line, in the ſame Proportion, as the firſt Line cuts the ſecond; that is, SR : RD :: FD : DC.

DEMONSTRATION.

Draw PC parallel to SD, and let $CP=r$, $RS=c$, $FS=d$, $RD=p$, $PS=m$, $FD=g$, and $DC=b$; then $c:r::d^2:\overline{d+m}^2::$ (by ſimilar Triangles) $g^2:\overline{g+b}^2$, alſo (by ſimilar Triangles) $r:g+b::c+p:g$; therefore $\frac{cg^2+2cgb+cb^2}{g^2}=r=\frac{cg+cb+pg+pb}{g}$, and conſequently $cgb+cb^2=pg^2+pgb$, or $cb\times\overline{g+b}=pg$

 $\times\overline{g+b}$

$\times \overline{g+b}$; therefore $cb = pg$, or $c : p :: g : b$; that is, SR : RD :: FD : DC. Q.E.D.

PROPOSITION XXIV.

As the Abſciſſa, is to any Diameter terminated by the Ordinate, ſo is the Square of that Ordinate, to the Rectangle of the Parts of the Ordinate made by that Diameter; that is, VG : OB :: $\overline{FG}^2$: FB × BC.

DEMONSTRATION.

Let OB $= m$, FB $= c$, BC $= r$, MO $= b$; then (by Prop. 21) $x : b :: y^2 : c^2$, and (by the laſt Prop.) $b : m :: c : r$; therefore $\frac{c^2 x}{y^2} = b = \frac{mc}{r}$, and $rcx = my^2$, or $x : m :: y^2 : rc$; that is, VG : OB :: $\overline{FG}^2$: FB × BC. Q.E.D.

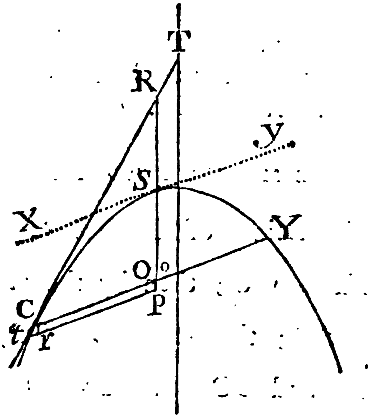
COROLLARY.

Hence it is manifeſt, OB : RD :: FB × BC : FD × DC; becauſe (by this Prop.) OB : VG :: FB × BC : $\overline{FG}^2$, and RD : VG :: FD × DC : $\overline{FG}^2$; conſequently OB : RD :: FB × BC : FD × DC.

PROPOSITION XXV.

If a Tangent cut any Diameter produced, and from the Point of Contact, an Ordinate be drawn to that Diameter; then the Diſtance (in the Diameter

Diameter produced) between the Vertex and Interfection of the Tangent, will be equal to the Abfciffa of the Ordinate; that is, $RS = RO$.



D E M O N S T R A T I O N.

Let $OS = x$, $Cr = OP = n$, $tr = m$, $RS = a$; $OC = y$, and then Pt, which is fuppofed to be indefinitely near to OC, will be $= m + y$, and $SP = n + x$; whence, by fimilar Triangles, $m : n :: y : n + x$; therefore $a + x = \frac{n \times y}{m}$, and (by Prop. 11.) $p \times SO = \overline{OC}^2$, alfo $p \times SP = \overline{Pt}^2$; that is, $px = y^2$ and $pn + px = m^2 + 2my + y^2$; therefore $px = y^2 + 2my - pn$; confequently $y^2 = y^2 + 2my - pn$, or $2my = pn$, and $n = \frac{2my}{p}$; let this Value be fubftituted for n in the firft Equation, and it will become $x + a = \frac{2my}{p} \times \frac{y}{m} = \frac{2y^2}{p} = \frac{2px}{p}$

G $= 2x$;

$= 2x$; consequently $a = 2x - x = x$; or RS = SO.

PROPOSITION XXVI.

If a Diameter be drawn from the Interſection of any two Tangents, it will biſect the Line, which joins the Point of Contact.

DEMONSTRATION.

From the Points of Contact Y, C, draw the Ordinates YO, CO; then by the laſt RS = SO and RS = SO; therefore SO = SO, and conſequently YO and CO being Ordinates to the ſame Diameter and Abſciſſa are equal, and in the ſame right Line. Q. E. D.

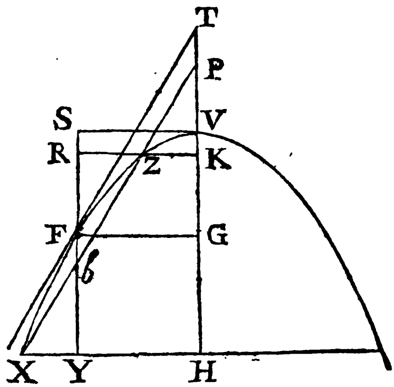
COROLLARY.

Hence another Method of drawing Tangents to a Parabola, from any Point without the Curve, may be deduced; for from the given Point R, draw the Diameter RP, and take therein, from the Vertex S, the Abſciſſa SO equal to the external Part RS; then through the Extremity of the Abſciſſa O, draw the right Line CY parallel to the Tangent XY, at the Vertex of that Diameter; and the Extremities of that Line will be Points in the Curve, in which Lines drawn from the given Point, will touch the Curve.

PROPOSITION XXVII.

If, from the Extremity of any Ordinate (Xb) to a Diameter, a right Line (as XY) be drawn at right

right Angles to the Diameter; then the Diſtance (XY) in that Line from the Extremity of the Ordinate to the Diameter, will be a Mean-proportional between the Parameter of the Axe, and the Abſciſſa of that Ordinate; that is, $p : XY :: XY : Fb$.



D E M O N S T R A T I O N.

Put $XY = a$, $Fb = X$; then (by Prop. 1.) $\overline{FG}^2 = px$, and (by Prop. 5.) $\overline{GT}^2 = 4x^2$; therefore (by 47. *Eu.* 1.) $\overline{FT}^2 = px + 4x^2 = \overline{p + 4x} \times x$; but (by Prop. 12.) $\overline{Xb}^2 = \overline{p+4x} \times X$, and (from ſimilar Triangles) $\overline{FT}^2 : \overline{FG}^2 :: \overline{Xb}^2 : \overline{XY}^2$; that is, $\overline{p+4x} \times x : px :: \overline{p+4x} \times X : a^2$; therefore $a^2 = px$ and $p : a :: a : X$, or $p : XY :: XY : Fb$.

Q. E. D.

The End of the PARABOLA.

A

COLLECTION

OF

PROBLEMS,

TO BE

Anſwered in the next NUMBER.

PROBLEM I. R.

A Certain Number of Perſons agreed to give Sixpence a-piece to a Waterman, to carry them from *London* to *Graveſend*, on Condition, that if others were taken in by the Way, they ſhould pay the ſame Price, and that of the Money thence ariſing, one Half ſhould go to the Waterman, and the other Half to be equally divided among the firſt Perſons; now they happened to take in, one fourth Part of their Number, and three over, by Means whereof they only paid Fivepence each; What Number were there in Company at firſt?

PRO-

PROBLEM II. *D.*

If a Cubic Foot of dry *English* Oak be put into a sufficient Quantity of fresh Water, how much of it will be immerged, and how much emerge, and what Weight must be laid on it, to make it level with the Water's Surface?

PROBLEM III. *L.*

How high must an Eye be elevated above the Earth's Surface, to see two fifth Parts thereof; allowing the Earth to be spherical, and its Circumference 25020 Miles?

PROBLEM IV. *T.*

Having Occasion to take the Declination of a Wall, and not being provided with proper Instruments for taking Altitudes, I had Recourse to the following Method: I observed, by laying my Eye close to the Edge of the Wall, the exact Time that the fixed Star *Pollux* came into the Plane of the Wall; and 56′ 10″ after (according to the Time shewn by a good Pendulum Clock) observed *Sirius* to pass through the Plane. From these Observations the Plane's Declination may be determined, and is here required; the Latitude of the Place being 51° : 10′ North.

PROBLEM V. *E—— H——.*

In what Latitude North may an erect declining Dial be erected South Easterly; the Stile's Height; the Distance of the Substile from the Meridian, and the Plane's Declination whereof are equal to each other? 'Tis also required to know what Hours on *May* the 10th, 1745, the Sun comes on, and goes off the Plane.

PRO-

PROBLEM VI. *I—— B——.*

At a certain Place in North Latitude, the Sun was observed to rise exactly at 3 H. 58 Min. and at Six o'Clock his Altitude was taken the same Morning, and found to be 15° : 20′, his Declination being then North; required, the Latitude where, and Day of the Year when, these Observations were made.

PROBLEM VII. *R.*

Two Circles touching each other inwardly being given; to describe a third Circle, that shall touch both the former, and also the right Line passing through their Centers.

PROBLEM VIII. *R.*

To draw a Chord, through a given Point, within a given Circle; the Parts whereof intercepted by that Point and the Periphery, may obtain a given Ratio.

PROBLEM IX. *L.*

The two Sides of any plane Triangle, and a right Line drawn from the Vertex, to the Middle of the Base being given; to determine the Triangle.

PROBLEM X. *N.*

The Perpendicular, the Line bisecting the vertical Angle, and the Line bisecting the Base of any plane Triangle being given; to determine the Triangle.

PROBLEM XI. *N.*

The Line bisecting the Base; the vertical Angle, and the Angle which the said bisecting Line makes with the Base being given; to describe the Triangle.

PROBLEM XII. *John Turner, London.*

To find a Point, in the Side of a given Square produced, from whence if a Line be drawn to the opposite

opposite Angle, the Part thereof intercepted between that Point, and the nearest Side of the Square, shall be of a given Length; to be constructed and demonstrated geometrically.

Problem XIII. *R.*

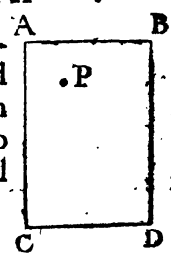
To determine a Point C, in a given right Line DF, from whence if two other right Lines AC and BC be drawn, to two given Points A and B, they shall comprehend an Angle ACB equal to a given Angle D.

Problem XIV. *R.*

If from the Extremities of the Base of any plane Triangle, two right Lines be drawn, intersecting each other in the Perpendicular, and terminating in the opposite Sides; right Lines drawn from thence, to where the Perpendicular meets the Base, will make Angles with the Base equal to each other; *Quære*, the Demonstration.

Problem XV. *S——A——.*

Let ABCD represent a given rectangular Billiard Table; it is required to find in what Direction, a Ball from the given Point P must be struck; so that after three Reflections, it shall fall into the Purse at the Angle B.



Problem XVI. *R.*

The Difference of the Diameters of two Circles (ABE and BCD) touching each other inwardly (in B) being given; it is required to draw from (A) the other Extremity of the greater Circle, a Tangent to the lesser, terminating in the Periphery of the greater; the Part whereof (DE) intercepted between the Point of Contact (D) and that Periphery may be of a given Length.

PRO-

PROBLEM XVII. *T.*

To draw a right Line through a given Point, terminating in two right Lines given by Position, so as to be the shortest possible.

PROBLEM XVIII. *D.*

The Sum of the Diameters of two Circles touching each other outwardly being given, also the Length of the mixed Line circumscribing them both; to find the Diameter of each.

PROBLEM XIX. *John Turner, London.*

If upon the principal Axe of any Curve, defined by the Equation $y = \frac{\overline{ax - x^2}|^{\frac{3}{4}} \times a^5 x^2}{\overline{a+x}|^9}$, a Semicircle be described; it is required, to exhibit in finite Terms, the exact Ratio between that Semicircle, and the whole curvilineal Space included between the said Curve and its Axis.

PROBLEM XX. *T.*

If the Earth be supposed to revolve round its Axis in 38 Minutes, and by Means thereof be projected two Bodies from off its Surface; the one in Latitude 52 Degrees, and the other under the Equator, with Velocities respectively as the Places from whence they are projected; it is required to determine the Trajectory of the former; and if it returns, its periodic Time, and where its Revolution will end; and likewise how far the latter will be from the Earth's Center in 6 Hours Time, allowing the Earth to be spherical, its Circumference 25020 *English* Miles.

N. B. This Problem was proposed in the *Ladies Diary* for the Year 1736; the Solution whereof several attempted, in the succeeding Diary; but had not the desired Success.

The END.

THE

Mathematician.

DISSERTATION II.

Upon the Progreſs *and* Improvement *of* GEOMETRY.

THE former Diſſertation in Nº I. having been concluded a little too abruptly, becauſe at the Time of its Publication we thought to ſay no more of it; but its having met with a candid Reception, has made us think it expedient to reſume and purſue the Subject more minutely; and particularly with a Regard to the Improvements that Geometry has received from thoſe its illuſtrious Sons, *Wallis*, *Barrow*, *Newton*, &c.

Since the hiſtorical Part of the Invention of Fluxions has been treated of by ſeveral Hands, we ſhall in the following Pages, after mentioning the

various Steps previous and preparatory thereto, endeavour to give as plain and clear an Account of that admirable Doctrine as the Nature of the Thing and our Abilities will admit, in order to evince, that it is truly and properly ſcientific; as will appear from the Accuracy and Reaſonableneſs of its Principles, and the Juſtneſs of its Rules, notwithſtanding it has been of late ſo much controverted.

To what has already been ſaid of Dr. *Wallis*'s Improvements may be added, that he found his Method of ſumming up Series's, to fail him in the Caſe of thoſe Progreſſions, whoſe Terms were the Roots of the Sums or Differences of ſimple Terms, called Roots univerſal, ſuch as $\sqrt{r^2 - 0a^2}$, $\sqrt{r^2 - 1a^2}$, $\sqrt{r^2 - 4a^2}$, $\sqrt{r^2 - 9a^2}$, $\sqrt{r^2 - 16a^2}$, &c. $\sqrt{r^2 - r^2}$, which he calls a Series of Terms in the ſubduplicate Ratio of a Series of Equals, leſſened by a Series of Secundans or Squares: Where, if r ſtand for the Radius of a Circle, and a for any of the indefinitely ſmall and equal Diſtances of the Ordinates in the Quadrant of the Circle, beginning at its Center, ſuch Quadrant is equal to the Sum of all the Terms of this Progreſſion; as he ſhews in Prop. 121 *Arith. Infinit.* They being the right Sines of which the Quadrant is compoſed; each of which are known to be a mean Proportional between $\overline{r + 0a}$ and $\overline{r - 0a}$, and between $\overline{r + 1a}$ and $\overline{r - 1a}$, and between $\overline{r + 2a}$ and $\overline{r - 2a}$, and ſo on. The ſame Series with the Sign of the ſecond Term under the Vinculum, changed into its Oppoſite, that is, $\sqrt{r^2 + 0a^2}$, $\sqrt{r^2 + 1a^2}$, $\sqrt{r^2 + 4a^2}$, $\sqrt{r^2 + 9a^2}$, *&c.* to $r^2 + r^2$, being ſummed up, would give the Quadrature of the equilateral Hyperbola, ſuppoſing r to ſtand for the ſemitranſverſe Diameter, and a any of the indefinitely

definitely ſmall and equal Diſtances on the Aſſymptote, from the Center of the tranſverſe Diameter, then, as is evident from Conics, the Ordinates compoſing the hyperbolic Space will be expreſſed by the laſt Progreſſion. He likewiſe ſhews two other Series's in Prop. 165, of the ſame Book; by the ſumming up of whoſe Terms, the Quadrature of the Circle and Hyperbola would be found, but labouring under the ſame Inconveniency. Here the Doctor ſtuck, and owns *Hic Labor, hoc Opus eſt.* However, being very ſolicitous to do ſomething towards the Quradrature of the Circle, which was his principal View when he engaged in the Proſecution of theſe Enquiries, (as he tells us in the Preface of that Work) he thought upon another Method, which he calls the Interpolation of Series; by which he means, a Method of diſcovering certain intermediate Terms of a regular Series or Progreſſion, by conſidering the Properties of the Progreſſion, and the Relations of the Terms to each other. Of this he gives ſeveral Inſtances for finding the Area of the Circle; whereof this is one: In the Progreſſion $1, \frac{3}{2}, \frac{15}{8}, \frac{105}{48}$, *&c.* Whoſe Terms are produced by the continual Multiplication of $1 \times \frac{3}{2} \times \frac{5}{4} \times \frac{7}{6}$, *&c.* to find the intermediate Term betwixt 1 and $\frac{3}{2}$. But the Reſult of his Enquiry was, that the Value of ſuch intermediate Term cannot be adequately expreſſed by any received Way of Notation; which is nothing more ſtrange than that $\sqrt{2}$, or any other ſurd Root, is not capable of being adequately expreſſed in that Way: But ſince the Value of $\sqrt{2}$, or any other Surd, may be expreſſed approximately by the common Notation; ſo likewiſe he found, that the approximate Value of the Square, or the Circle's Area, was $1 \times \frac{3 \times 3 \times 5 \times 5 \times 7 \times 7, \&c.}{2 \times 4 \times 4 \times 6 \times 6 \times 8, \&c.}$ *in infinitum.* Notwithſtanding this Diſappointment, Dr. *Wallis*

 opened

opened the Gate into a wide Field of geometrical Knowledge: For, in his arithmetical Works, published in 1657, he first reduced the Fraction $\frac{A}{1-R}$ by a perpetual Division into an infinite Series $A+AR+AR^2+AR^3+AR^4$, *&c.*

Mr. *James Gregory*, in his *True Quadrature of the Circle and Hyperbola*, published in the Year 1667, gave a converging Series, or at least shewed the Constitution of a converging Series, which proceeds by Pairs of Terms; such, that the Difference between any Pair is greater than the Difference between the next subsequent Pair; and that after the same Manner that the second Pair is formed by analytical Operation from the first, in like Manner is the third formed from the second, and so on; by which the Difference continually lessening, becomes less than any given Quantity; and the Series being supposed to be continued *in infinitum*, that Difference quite vanishes, and the two Terms become equal, either of which is the Quantity sought; whereby you may approach to the Area of the Hyperbola, as well as that of the Circle, as near as you please.

In *April*, next Year, Lord Viscount *Brounker* published an infinite Series for the same Purpose, which is that called the *Newtonian*, being now generally used; where the Aggregate of all the Terms, infinite in Number, is equal to the Quantity sought; and the greater the Number of them taken from the Beginning, is, the nearer doth their Aggregate approach to the Quantity sought. The same Year Mr. *Nicholas Mercator*, an *Holsatian* by Birth, but who had spent a great Part of his Life in *England*, published his *Logarithmotechnia*; in which he shewed how Lord Viscount *Brounker*'s Series might be found, by reducing a complex Fraction to an infinite Series of simple Terms by

Di-

Division.; which was but a small Improvement upon what Dr. *Wallis* had shewn in his *Math. Univ.* cap. 33. with respect to Division. *Mercator* then had no more to do but to apply the same Person's Method of Quradratures in his *Arith. Inf.* for squaring those several simple Terms. Upon the Publication of the *Logarithmotechnia* Dr. *Wallis* illustrated the Discovery; and gave another infinite Series for the same Purpose, in the *Philosophical Transactions* for *August* 1668. Towards the End of the same Year, Mr. *James Gregory* published his *Exercitationes Geometricæ*; in which he promoted and enlarged *Mercator*'s Discovery, and gave a geometrical Demonstration of it, by Means of summing up the Secants of a circular Arch.

We must not omit to observe a very remarkable Conclusion that Dr. *Wallis* obtained in his *Arith. Inf.* by his gradual Method of Induction. He considered those Progressions as consisting of an infinite Number of Terms; and having adopted *Cavalerius*'s Method of Indivisibles, the Elementa of which geometrical Figures were by that Method composed, were naturally represented by the Terms of those Progressions, *viz.* the last Term, which represented the lowest Ordinate of a Curve, being still finite; and the intermediate Terms from o to the last, being infinite in Number, represented Ordinates applied to the Axis at infinitely small and equal Distances, betwixt the Vertex and lowest Ordinate; or, perhaps, these Terms represented any other Lines, right or curved; or any plain or curved Surfaces, in the Case of Solids, which were proportional to them: Now nothing was wanted but a Method of summing them up. At last, the Doctor found this most general and comprehensive Proposition, which contains the Substance of his whole Book, *viz.* The Sum of all the Terms of

any

any Series of the (m) Powers of Quantities in arithmetical Progreſſion from o, is equal to the Product of the laſt Term, by the Number of Terms; and this divided by the Index (m) plus 1. Which amounts to this; ſuppoſing 0, 1, 2, 3, 4, &c. x, to be an arithmetical Progreſſion, conſiſting of an infinite Number of Terms, in the natural Order of Numbers, having the laſt Term x; and let 0^m, 1^m, 2^m, 3^m, 4^m, &c. x^m, be a Progreſſion of Terms, which are any the ſame Power, Root, or Dimenſion, whatſoever of the former Terms, whoſe Exponent is denoted by (m); then ſhall the Sum of this laſt Series or Progreſſion of Powers, Roots, &c. be equal to $\frac{x^{m+1}}{m+1}$. Here Dr. *Wallis* may be ſaid to end, and Sir *Iſaac Newton* to begin his Improvements; for this is in Subſtance the ſame with his firſt Rule in his Analyſis, and upon which he builds his Quadratures of Curves. *Wallis* demonſtrated this general Propoſition by an Induction from ſeveral particular ones, collected at laſt into one by a Table of Caſes: But Sir *Iſaac* reduced all the Caſes into one, and demonſtrated it univerſally by Means of an indefinite Index, which he firſt introduced into analytical Operations.

When theſe Diſcoveries of Dr. *Wallis* (which were indeed very noble and uſeful, and, in Point of Generality and Extent, far exceeding every Thing that had been before done by others in the Geometry of Curve Lines) were made public, it was objected to him by Mr. *Fermat* and others, that however valuable his Diſcoveries were, and true in themſelves, yet the Demonſtrations of them in the Way of Induction, did not come up to that Accuracy which a geometrical Subject required, and which the ancient Geometricians had all along obſerved in their Performances. To which Dr. *Wallis*

Wallis made Anſwer, (even as he had remarked in the *Arithmetic of Infinites* itſelf) That he did not ſo much deſign to demonſtrate his Diſcoveries, as to lay open to others the Method he uſed in making them, which the Antients purpoſely concealed; that, notwithſtanding, he thought the Proof, by Way of Induction, was ſatisfying and convincing; that it would be an eaſy Matter for any Perſon, moderately ſkilled in Geometry, to demonſtrate theſe Things, with all the Pomp and Apparatus made uſe of by the Antients; but that was a Labour he never deſigned to undertake. However, to give ſome Satisfaction in this Matter, he ſhews, by ſome Examples, in the 78th Chapter of his Algebra, how the Propoſitions he had diſcovered might be demonſtrated after the Manner of the Antients, in Imitation of what had been done by *Archimedes* in the 10th and 11th Propoſitions of his Treatiſe of ſpiral Lines; wherein *Archimedes* demonſtrates what is the Sum of all the Terms of a Progreſſion of Squares, whoſe Sides conſtitute an arithmetical Progreſſion of Lines, having the common Difference equal to the firſt Term, when compared with ſo many Times the greateſt Square; and the Limits betwixt which the Sum of the Terms of ſuch a Progreſſion is contained, although the common Difference of their Sides be not the ſame as the firſt of them; which he applies to the finding the Relation of the ſpiral Spaces to the circular Sectors; even as Dr. *Wallis*, by proſecuting this Affair to a much greater Length, ſhews how to find not only the Sum of all the Squares, but the Sum of any Powers or Roots whatſoever of an arithmetical Progreſſion; and thereby to compare infinite Numbers of curvilinear Areas with right-lined Figures, and with one another. And truly, when one attentively conſiders this elaborate Treatiſe of *Archimedes*, and the other Works of that

that ſubtle and penetrating Genius, one cannot help ſeeing Dr. *Wallis's Arithmetic of Infinites*, Mr. *Gregory's Method of Inſcription and Circumſcription of Polygons*, and even Sir *Iſaac's Method of prime and ultimate Ratio's*, beginning to diſcover themſelves, as it were, in Embryo, in order to be brought forth afterwards to open Light and Perfection.

If it ſhould be here objected, that ſince all the Terms of an infinite Series are unaſſignable, it is impoſſible to determine their Sum; to obviate this, let it be conſidered, that a Number actually infinite, (*i. e.* all whoſe Units can be actually aſſigned, and yet is without Limits) is a plain Contradiction to all our Ideas about Numbers; for whatever Number we can actually conceive, or have any proper Idea of, is and muſt be always determinate and finite; ſo that a greater after it may be aſſigned, and a greater after this, and ſo on, without a Poſſibility of ever coming to an End of the Addition or Increaſe of Numbers aſſignable; which Inexhauſtibility or endleſs Progreſſion in the Nature of Numbers, is all that we can diſtinctly underſtand by the Infinity of Numbers; and therefore to ſay, that the Number of any Things is infinite, is not ſaying, that we comprehend their Number, but indeed the contrary; the only Thing poſitive in this Propoſition being this, *viz.* that the Number of theſe Things is greater than any Number which we can actually conceive and aſſign. We eaſily conceive, that a finite Magnitude may become greater and greater without End, or that no Termination or Limit can be aſſigned of the Increaſe which it may admit; but we do not therefore clearly conceive Magnitude increaſed an infinite Number of Times. Mr. *Locke*, who wrote his excellent Eſſay, that we might diſcover how far the Powers of the Underſtanding reach, to what Things they are in any De-

Degree proportionate, and where they fail us, he acknowledges, that we easily form an Idea of the Infinity of Number; to the End of whose Addition there is no Approach: But he distinguishes between this and the Idea of an infinite Number; and subjoins, that how clear soever our Idea of the Infinity of Number may be, there is nothing more evident than the Absurdity of the actual Idea of an infinite Number. He likewise observes, that whilst Men talk and dispute of infinite Magnitudes, as if they had as compleat and positive Ideas of them as they have of the Names they use for them, or as they have of a Yard, an Hour, or any other determinate Quantity, it is no Wonder, if the incomprehensible Nature of the Thing they discourse of, or reason about, leads them into Perplexities and Contradictions, and their Minds be overlaid by an Object too large and mighty to be surveyed and managed by them. Mathematicians, indeed, abridge their Computations by the Supposition of Infinites; but when they pretend to treat them on a Level with finite Quantities, they are sometimes led into such Doctrines as verify the Observation of this judicious Author.

We cannot apply to an infinite Series the common Notion of a Sum, *viz.* a Collection of several particular Numbers, that are joined and added together, one after another; for this supposes, that these Particulars are all known and determined; whereas, the Terms of an infinite Series cannot be all separately assigned, there being no End in the Numeration of its Parts, and therefore it can have no Sum in this Sense. But again, consider, that the Idea of an infinite Series consists of two Parts, *viz.* the Idea of something positive and determined, in so far as we conceive the Series to be actually carried on; and the Idea of an inexhaustible Remainder still behind, or an endless

Addition of Terms that can be made to it, one after another; hence we may conceive it as a Whole of its own Kind, and therefore may be ſaid to have a real Value, whether that be determinable or not. Now, in ſome infinite Series, this Value is finite or limited, *i. e.* a Number is aſſignable, beyond which the Sum of no aſſignable Number of Terms of the Series can ever reach, nor indeed ever be equal to it; yet may approach to it in ſuch a Manner, as to want leſs than any aſſignable Difference; and this we may call the Value or Sum of the Series; not as being a Number found by the common Method of Addition, but as being ſuch a Limitation of the Value of the Series, taken in all its infinite Capacity, that if it were poſſible to add them all one after another, the Sum would be equal to that Number. Again, in other Series, the Value has no Limitation; and we may expreſs this by ſaying, The Sum of the Series is infinitely great; which indeed ſignifies no more than that it has no determinate and aſſignable Value; and that the Series may be carried ſuch a Length, as its Sum, ſo far, ſhall be greater than any given Number. In ſhort, in the firſt Caſe, we affirm, there is a Sum; yet not a Sum taken in the common Senſe; in the other Caſe, we plainly deny a determinate Sum in any Senſe. According to the common Rule for ſumming up a finite Progreſſion of a geometric Series decreaſing, where r is the Ratio, l the firſt Term, and A the leaſt, the Sum is $= \frac{rl - A.}{r - 1}$ If we ſuppoſe A the leſſer Extream actually decreaſed to 0, then the Sum of the whole infinite Series is $= \frac{rl}{r-1}$: For it is demonſtrable, that the Sum of no aſſignable Number of Terms of the Series can ever be equal to that Quotient; and yet no Num-

Number less than it, is equal to the Value of the Series. And whatever Consequences follow from the Supposition of $\frac{rl}{r-1}$, being the true and adequate Value of the Series, taken in all its infinite Capacity, as if the Whole were actually determined and added together, can never be the Occasion of any assignable Error, in any Operation or Demonstration, where it is used in that Sense; because, if you say it exceeds that adequate Value, yet it is demonstrable, that this Excess must be less than any assignable Difference, which is in Effect no Difference; and so the consequent Error, will be in Effect no Error: For if any Error can happen from $\frac{rl}{r-1}$ being greater than it ought to be, to represent the compleat Value of the infinite Series, that Error depends upon the Excess of $\frac{rl}{r-1}$ over that compleat Value; but this Excess being unassignable, that consequent Error must be so too; because still the less the Excess is, the less will the Error be that depends upon it. For which Reason, we may justly enough look upon $\frac{rl}{r-1}$ as expressing the adequate Value of the infinite Series: But we are farther satisfied of the Reasonableness of this, by finding, in Fact, that a finite Quantity does actually convert into an infinite Series, as appears in the Case of infinite Decimals. *E. g.* $\frac{2}{3} = .6666$, *&c.* is plainly a geometric Series in the Decimal Scale, from $\frac{6}{10}$ in the continual Ratio of 10 to 1; and may very justly be expressed by an infinite Series of the Reciprocals of the Powers of 10, which is the Root of the Scale; and if 10 be denoted by x, will be $6x^{-1} + 6x^{-2} + 6x^{-3} + 6x^{-4}$, *&c.* And, reversely, if we take

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this Series, and find its Sum by the preceding Expreſſion, it comes to the ſame $\frac{2}{3}$; for $l = 6x^{-1} = \frac{6}{10}$, $r = x = 10$, therefore $rl = \frac{60}{10} = 6$; and $r - 1 = 9$, whence $\frac{rl}{r-1} = \frac{6}{9} = \frac{2}{3}$. By the ſame Artifice, uniformly carried on, may all Decimals, ſimple or mixed, be expreſſed, provided we aſſume the Co-efficients 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, as Occaſion ſhall require, to its proper Term in the Series; thus the mixt Number 526.384, becomes $5x^{2} + 2x^{1} + 6x^{0} + 3x^{-1} + 8x^{-2} + 4x^{-3}$.

The like may be done by any other Scale as well as the Decimal Scale, or by admitting any other Root beſides 10, to be the Root of our Arithmetic; for the Root 10 was an arbitrary Number, and, at firſt, aſſumed by Chance, without any previous Conſideration of the Nature of the Thing. Other Numbers, perhaps, may be aſſigned, which would have been more convenient, and which have a better Claim for being the Root of the vulgar Scale of Arithmetic. The Sexagenary or Sexageſimal Scale obtains among Aſtronomers; and expreſſes all poſſible Numbers, Integers, or Fractions, rational or ſurd, by the Powers of 60; and certain numeral Co-efficients, not exceeding 59. Any Number whatever, whether Integer or Fraction, may be made the Root of a particular Scale of Arithmetic, and all conceiveable Numbers may be expreſſed or computed by that Scale, at leaſt, by Approximation, admitting only of integral and affirmative Co-efficients, whoſe Number (including the Cypher 0) need not be greater than the Root. And it appears by the Inſtance above, that ſome Numbers may be expreſſed by a finite Number of Terms in one Scale, which in another cannot be expreſſed but by Approximation, or by a Progreſſion of Terms *in infinitum*. It further appears, that a Number computed by any one Scale, is eaſily re-

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duced to any other Scale assigned, by substituting instead of the Root in one Scale, what is equivalent to it expressed by the Root of the other Scale. Thus to reduce Sexagenary Numbers to Decimals; because 60, or X is $=6\times10$; or $X=6x$ therefore $X^2=36x^2$, $X^3=216x^3$, *&c.* by the Substitution of these we shall easily find the equivalent Decimal Number. All vulgar Fractions, and mixed Numbers, are, in some Measure, the Expressions of Numbers by a particular Scale; or making the Denominator of the Fraction to be the Root of a new Scale; thus $\frac{2}{3}$ is in Effect $0\times3^0+2\times3^{-1}$, and $8\frac{3}{5}$ is the same as $8\times5^0+3\times5^{-1}$, and so of others.

The Co-efficients in these Scales are not necessarily confined to be affirmative integer Numbers less than the Root, (though they should be such, if we would have the Scale to be regular;) but as Occasion requires, they may be any Numbers whatsoever, affirmative or negative, Integers or Fractions. And, indeed, they generally come out promiscuously in the Solution of Problems. Nor is it necessary, that the Indices of the Powers should be always integral Numbers; but may be any regular arithmetical Progression whatever; and the Powers themselves either rational or irrational. For, supposing the Root of the Scale to be an indefinite or general Number, represented by x or y, &c. and assuming the general Co-efficients a, b, c, d, &c. which are Integers or Fractions, affirmative or negative, as it may happen; we may form such a Series as this, $ax^4+bx^3+cx^2+dx^1+ex^0$; which will represent some certain Number expressed by the Scale, whose Root is x. If such a Number proceeds *in infinitum*, then it is truly and properly called an infinite or converging Series; x being then supposed greater than Unity. Such, for Example, is $x+\frac{1}{2}x^{-1}+\frac{1}{3}x^{-2}+\frac{1}{4}x^{-3}$, *&c.*

&c. And it may have any deſcending arithmetical Progreſſion for its Indices, as $x^m + \frac{2}{3}x^{m-1} + \frac{3}{4}x^{m-2} + \frac{4}{5}x^{m-3}$, *&c.*

And thus we have been led by proper Gradations, that is, by arguing from what is well known and commonly received, *viz.* the Doctrine of decimal Fractions; to what before appeared to be difficult and obſcure, *viz.* the Knowledge of an univerſal or infinite Series. The great Similitude between the Nature and Operations of them, occaſions the former to be a convenient Illuſtration of the latter; for the chief Difference between theſe infinite Series's in decimal Arithmetic, and thoſe in the literal or ſpecious, is, that in the former there is only one Scale or Progreſſion of Terms, which varies in a decuple Ratio; and the Co-efficients are all poſitive Integers below 10; whereas, in the latter, which is of a more general and indefinite Nature, the Scales or Progreſſions may be infinitely varied, in a decuple, or any other Ratio whatever; and the numeral Co-efficients may be any Numbers integral or fractional, poſitive or negative, as hath been before obſerved. It is likewiſe evident, that the Operations in decimal Arithmetic, by which the Quotient or Quantity ſought is diſcovered, correſpond to the Operations relating to infinite Series in ſpecious Arithmetic; in this Reſpect, that every new Step of the Operation, in both Caſes, by which the conſtituent Parts of the Quotient are found, makes as great an Advance towards the Supply of what the Quotient is yet deficient, as the Nature of the Progreſſion will admit. But all general Series, which are commonly the Reſult in the higher Problems, muſt paſs by Subſtitution to particular Scales or Series; and theſe, when applied to Practice, muſt have their indefinite and determinate Characters, in order to be finally reduced to the Decimal Scale. And the

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Art of finding ſuch general Series's, and then their Reduction to particular Scales, and laſt of all to the common Decimal Scale, is almoſt the Whole abſtruſer Parts of Analytics.

Our having dwelt ſo long upon the Nature of Series's here, will have its Uſe, by contracting what is hereafter to be ſaid, when we come to treat of Sir *Iſaac Newton*'s analytical Improvements, which were the Keys of his Operations in the ſublime Geometry.

So much for Dr. *Wallis*. The next Promoter of Geometry, with reſpect to Time, among our Countrymen, was Dr. *Barrow*, a Man of a penetrating Genius, and very indefatigable: He had amaſſed a large Magazine of Learning; and his general Character was, that whatever Subject he treated on, he exhauſted: He was a perfect Maſter of the ancient Geometry; and has obliged us with compendious, yet clear Demonſtrations of what is left of the geometrical Writings of *Euclid*, *Archimedes*, *Apollonius*, and *Theodoſius*. But the Advances he made in curve-lined Geometry, his own particular Improvements, are contained in his Lectures. He begins with treating on the *Generation of Magnitude*, which comprehends the Original of mathematical Hypotheſes. Magnitude may be produced various Ways, or conceived ſo to be; but the primary and chief among them is that performed by *Local Motion*, which all of them muſt in ſome Sort ſuppoſe; becauſe, without Motion, nothing can be generated or produced: So true is *Ariſtotle*'s Axiom, *viz.* He that is ignorant of Motion, is neceſſarily ignorant of Nature. What Mathematicians chiefly conſider in Motion are theſe two Properties, *viz.* The Mode of Lation, or Manner of Bearing; and the Quantity of the motive Force. From theſe Springs the Differences of Motions flow; but becauſe the Quantity of motive Force

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cannot be known without Time, the Doctor gives a long metaphysical Account of the Nature of Time; which he defines to be, abstractedly, *The Capacity or Possibility of the Continuance of any Thing in its own Being.* Towards the latter End whereof, he agrees with *Aristotle*, that we not only measure Motion by Time, but also Time by Motion; because they determine each other: For in like Manner as we first of all measure a Space by some Magnitude, and declare it is so much; and afterwards, by Means of this Space, compute other Magnitudes correspondent with it: So we first assume Time from some Motion, and afterwards judge thence of other Motions; which, in Reality, is no more than comparing some Motions with others, by the Assistance of Time; just as we investigate the Ratio's of Magnitude by the Help of some Space. *E. g.* He who computes the Proportion of Motion by the Proportion of Time, does no more than get the said Ratio of Motions from Clocks, Dials, or from the Proportion of solar Motions in the same Time. Again, because Time is a Quantity uniformly extended, all whose Parts correspond, either proportionally to the respective Parts of an equal Motion, or to the Parts of Spaces moved through with an unequal Motion; it may therefore be very aptly represented to our Minds, by any Magnitude alike in all its Parts; and especially the most simple ones, such as a strait or circular Line; between which and Time there happens to be much Likeness and Analogy: For as Time consists of Parts altogether similar, it is reasonable to consider it as a Quantity endowed with one Dimension only; whether we imagine it to be made up, as it were, either of the simple Addition of rising Moments, or of the continual Flux of one Moment; and for that Reason ascribe only Length to it, and determine its Quantity by the

the Length of a Line paſſed over. As a Line is looked on to be the Trace of a Point moving forward, being in ſome Sort diviſible by a Point, and may be divided by Motion one Way, *viz.* as to Length; ſo Time may be conceived as the Trace of a Moment continually flowing; having ſome Kind of Diviſibility from an Inſtant, and from a ſucceſſive Flux, inaſmuch, as it can be divided ſome Way or other. And like as the Quantity of a Line conſiſts of but one Length following the Motion, ſo the Quantity of Time purſues but one Succeſſion ſtretched out, as it were, in Lengrh; which the Length of the Space moved over, ſhews and determines. Time may therefore always be expreſſed by a right Line; firſt, indeed, taken or laid down at Pleaſure; but whoſe Parts will exactly anſwer to the proportionable Parts of Time, as its Points do the reſpective Inſtants of Time, and will aptly ſerve to repreſent them.

The Doctor next proceeds to the effective Force of Time, being the ſame as what he before called the Motive Force by which Magnitudes are generated. He conſiders this as a Kind of Quantity, capable of Computation, like other Quantities: For it is plain from Experience, that when two moveable Bodies depart from the ſame Place along the ſame Line, the one moves a greater Space than the other in the ſame Time; the Reaſon of which can only be this, that that Body which moves ſwifteſt, is acted upon by a greater Force or motive Power; this Force therefore admitting of greater and leſſer Modifications, may be juſtly conceived as diviſible into any infinite or indefinite Parts; the leaſt of which is called Reſt, or the loweſt Degree of Velocity: Conſidering therefore the Thing abſolutely, in order to repreſent the Quantity of this Force juſtly to the Mind, we need only lay down ſome regular Magnitude in its Stead.

As a right Line is the most simple and perspicuous of any, it is therefore the fittest to represent any Degree thereof. When this Force comes under a mathematical Consideration, it is called Velocity; which is defined to be *That Power by which a moveable Body can pass over a given Space in a given Time*; whence it follows, that every particular Quantity of any Velocity cannot be known, neither by the Space moved through only, nor by the Time singly, but may be found by Calculation from the Quantity of Space and Time together; as on the contrary, the Quantity of Time may be obtained from the Quantity of the Space and Velocity together: Nor does the Quantity of Space (so far as it can be known this Way by Motion) depend wholly upon the Quantity of a definite Velocity, or upon any assigned Time, but upon the conjoint Ratio of both. The Quantity of Space is found after the same Manner as we do that of a Superficies, by its Dimensions; but the Quantities of Velocity and Time are found exactly after the same Manner as when a Superficies and one of its Dimensions are given, we thereby find the other: For to every Moment of Time there answers some Degree of Velocity, which a moveable Body is then conceived to have; to which Degree some Length of the Space moved over answers. When Time flows equally, it will be most aptly represented by a right Line; and the several Degrees of Velocity, whether equal or unequal in each Instant, may be also expressed by right Lines; and because these Degrees of Velocity do in every Moment of Time pass over one another independently, and without Mixture; therefore, if right Lines parallel to each other, and horizontal, be drawn through all the Points of the perpendicular Line representing the Time, the plain Superficies thence resulting, will exactly represent the Aggregate of the

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Degrees of Velocity; which Superficies having its Parts proportionable to the reſpective Parts of the Space moved through, may very well repreſent that Space. If the Velocities anſwering to each Inſtant of Time are equal, this Superficies will be a Parallelogram; if unequal, a Triangle: From the Properties of the former Figure are deduced all the Theorems of equable and uniform Motion; and from the latter, all thoſe which concern equally accelerated Motion. Moreover, if the Degrees of Velocity, in a continual Succeſſion from Reſt, throughout every Inſtant of Time, to a given Degree, be conceived to increaſe to it, or decreaſe from thence to Reſt, in the Progreſſion of the ſquare Numbers, the aggregatical Velocity, as well as the Space moved through, may moſt conveniently be repreſented by the Semiparabola; whoſe Vertex denotes Reſt, the ſeveral equal Parts of the Abſciſs, the given equal Times, and the Ordinates, the reſpective Degrees of Velocity, from a well known Property of the Parabola. In like Manner, any ſuppoſed Degrees of Velocity, any how increaſing or decreaſing continually, or interruptedly after any imaginable Way, may be truly and conveniently expreſſed by right Lines applied to that repreſenting the Time, keeping whatever Proportion any one is pleaſed to aſſign; ſo that knowing from thence the Meaſure of the repreſentative Space, the Quantity of Space moved through will be eaſily had, and the contrary; for it is eaſy to deduce Theorems, if any one knows rightly and congruouſly how to reduce Quantities of any Kind ſoever, ſubject to his Contemplation, to analogous Magnitudes.

Perhaps, this dry Account of theſe metaphyſical Subjects may ſeem tedious; but if it be conſidered, that hereupon are founded the Theories of the Deſcent of heavy Bodies, of Pendulums, and of Projectiles; the reducing of which to geometrical

Demonſtrations raiſed the famous *Galileo* to ſo high a Reputation, that he was ſaid to have added two new Sciences to the Mathematics; and when we further conſider, that the Doctrine of Fluxions is compriſed in two mechanical Problems, and that Mechanics, or the Doctrine of Motion, depends upon Computation of the Quantities of Time, Velocities, and Forces, it will then appear, that theſe Conſiderations directly lead towards Fluxions, and that it cannot be Time ill ſpent to conſider their Nature abſtractedly; unleſs we could be content to know the Manner of operating by them only, without contemplating the Reaſon of them in Theory, and knowing whether they are really ſcientific or no.

To be continued.

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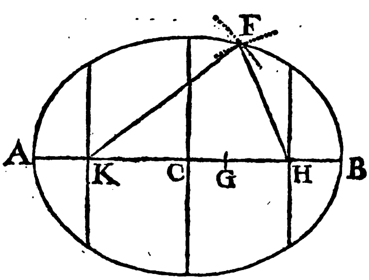
CONIC SECTIONS.

Of the ELLIPSE.

The GENESIS.

IF in any right Line, drawn upon a Plane, as AB, there be taken two Points K and H, equally remote from the Middle thereof, and in the same also be taken any other Point G, and from the former as Centers, with the Radii AG, BG respectively, be described two Areas; they will intersect each other in the Periphery of an Ellipse, passing through the Extremities of the Line AB.



For, by Construction, $KF = AG$, and $HF = GB$; therefore $HF + KF = AG + GB = AB$.

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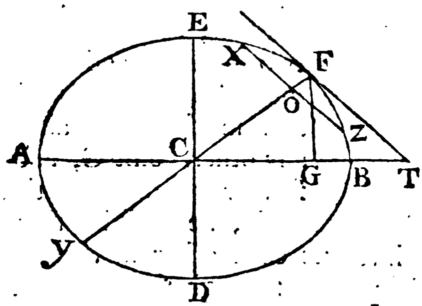
In like Manner an indefinite Number of Points may be found; through which, if a Curve Line be ſuppoſed drawn, it will comprehend a Space called an Ellipſe.

DEFINITIONS.

1. The Points H and K are called the Foci.
2. A Diameter is a right Line which paſſes through C, the Middle of AB, and biſects all Lines within the Curve, that are parallel to the Tangent touching its Vertex, and the Lines ſo biſected are called Ordinates to that Diameter; ſo FY is a Diameter; XO=OZ are Ordinates, being parallel to the Tangent touching the Curve in F, the Vertex of the Diameter.



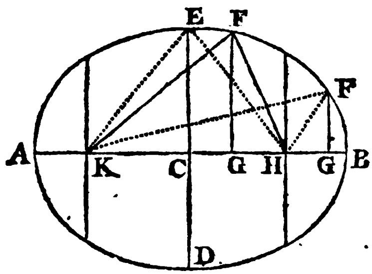
3. The Point of Interſection C, of all the Diameters, is called the Center.
4. That Diameter on which the Ordinates ſtand at right Angles, is called the tranſverſe Axe, as AB; and that which paſſes through the Center, cutting it at right Angles, is called the conjugate Axe, as ED.
5. The Point, where the Ordinates interſect the Diameter, is called the Point of Application, as G and O.

6. The

6. The Segment of the Diameter, intercepted between the Vertex and the Point of Application, is called the Abſciſſa; as FO, OY; or BG, AG.

PROPOSITION I.

As the Square of any Ordinate to the tranſverſe Axe is to the Rectangle of the Abſciſſas which it divides, ſo is the Square of the Conjugate to the Square of the tranſverſe Axe.



DEMONSTRATION.

Let $AC=t$, $CE=c$, $KC=b$, $CG=x$, $FG=y$, and z equal to the Difference between the Line KF and the Semi-tranſverſe Axe AC; then $KH=2b$, $KG=b+x$, and $GH=x-b$ or $b-x$, according as the Point G falls on this or that Side the Focus H; alſo, by the Geneſis, $KF=t+z$ and $FH=t-z$; whence (by *Eu.* 47. 1.) $\overline{HF}^2=\overline{HG}^2+\overline{GF}^2$; or $t^2-2tz+z^2=b^2-2bx+x^2+y^2$, and $\overline{KF}^2=\overline{KG}^2+\overline{GF}^2$;

or

or $t^2+2tz+z^2=b^2+2bx+x^2+y^2$; hence, the former of these Equations taken from the latter gives, $4tz=4bx$; therefore $z=\frac{4bx}{4t}=\frac{bx}{t}$; which being substituted in Place of z, in either of the foregoing Equations, there will come out $t^4+b^2x^2=t^2b^2+t^2x^2+t^2y^2$; but $b^2=t^2-c^2$, by the Genesis; therefore $\overline{t^2-x^2}\times c^2=t^2y^2$; which reduced to an Analogy, gives $y^2:\overline{t+x}\times\overline{t-x}::c^2:t^2$; that is, $\overline{FG}^2:AG\times GB::\overline{DE}^2:\overline{AB}^2$. Q. E. D.

Corollary.

Let any Abscissa be x, and its Ordinate y, the transverse Axis t, and the Conjugate c; (which Symbols represent the same Things in all the following Demonstrations) then by this Theorem, $t^2:c^2::\overline{t-x}\times x:y^2$; or $t^2y^2=c^2tx-c^2x^2$; which generally is called the Equation of the Curve.

Definition.

A third Proportional to the transverse and conjugate Axis, is called the Parameter of the Axe; that is, if for the Parameter be put p, then $t:c::c:p$; therefore $tp=c^2$.

Proposition II.

As the transverse Axe, is to its Parameter, so is the Rectangle of any two Abscissas, to the Square of the Ordinate which divides them.

De-

DEMONSTRATION.

By the Definition of the Parameter $tp = c^2$, and by putting tp in the Equation of the Curve for c^2, a new Equation of the Curve will be produced in Terms of the Parameter, &c. viz. $ty^2 = tpx - px^2$; or $y^2 = \frac{p}{t} \times \overline{tx - x^2}$; therefore $t : p :: \overline{t - x} \times x : y^2$. Q. E. D.

COROLLARY.

As the Rectangle of any two Abscissas, is to the Square of the Ordinate which divides them, so is the Rectangle of any other two Abscissas, to the Square of the Ordinate which divides them. For (by this Prop.) $\overline{t - x} \times x : y^2 :: t : p :: \overline{t - X} \times X : Y^2$.

PROPOSITION III.

The transverse Axe into one fourth of its Parameter, is equal to the Rectangle of the greatest and least Distance of either Focus from the Vertex; that is, $\frac{1}{4}p \times AB = AH \times HB = BK \times KA$.

DEMONSTRATION.

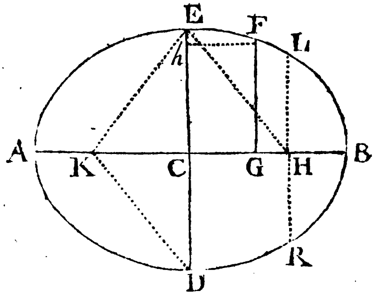
Let $HB = q$, then $HA = t - q$; and $CH = \frac{1}{2}t - q$: But $\overline{HE}^2 = \overline{EC}^2 + \overline{CH}^2$; that is, $\frac{1}{4}t^2 = \frac{1}{4}t^2 - tq + q^2 + \frac{1}{4}c^2$; or $\overline{t - q} \times q = \frac{1}{4}c^2 = \frac{1}{4}pt$; or $\frac{1}{4}p \times AB = AH \times HB$. Q. E. D.

COROLLARY.

The semi-conjugate Axe, is a mean Proportional between the greatest and least Distance of either Focus from the Vertexes: For since $\overline{t-q} \times q = \frac{1}{4}c^2$; therefore $t-q : \frac{1}{2}c :: \frac{1}{2}c : q$; that is, AH : CD :: CD : HB.

PROPOSITION IV.

The Parameter of the Axe, is double the Ordinate applied to the Focus.



DEMONSTRATION.

Let the focal Distance be q, and the Ordinate passing through the Focus y; then (by Prop. 2.) $t : p :: \overline{t-q} \times q : y^2$; but (by Prop. 3.) $\overline{t-q} \times q = \frac{1}{4}pt$; therefore $t : p :: \frac{1}{4}pt : \frac{1}{4}p^2 = y^2$, and $\frac{1}{2}p = y$; or $p = 2y$. Q.E.D.

PRO-

PROPOSITION V.

The Diſtance between the Foci, is a mean Proportional between the Sum and Difference of the Tranſverſe and conjugate Axe; that is, AB + DE : KH :: KH : AB — DE.

DEMONSTRATION.

For KH put b; then $\overline{KD}^2 - \overline{CD}^2 = \overline{KC}^2$; that is, $\frac{1}{4}t^2 - \frac{1}{4}c^2 = \frac{1}{4}b^2$; or $t^2 - c^2 = b^2$; therefore $t + c : b :: b : t - c$; or AB + DE : KH :: KH : AB — DE. Q. E. D.

PROPOSITION VI.

A fourth Proportional to the Conjugate, Tranſverſe, and any Ordinate, is equal to a mean Proportional between the Abſciſſas of that Ordinate.

DEMONSTRATION.

Let the fourth Proportional be b; then $c : t :: y : b$; therefore $b = \frac{ty}{c}$; but (by Prop. 1.) $t^2 : c^2 :: \overline{t - x} \times x : y^2$; therefore (by *Eu.* 22. 6.) $t : c :: \sqrt{\overline{t - x} \times x} : y$, and $\sqrt{\overline{t - x} \times x} = \frac{ty}{c} = b$. Q. E. D.

PROPOSITION VII.

The Diſtance between the Foci, is a mean Proportional between the tranſverſe Axe, and the Dif-

ference of the tranſverſe Axe and the Parameter; that is, AB : KH :: KH : AB — LR.

DEMONSTRATION.

Becauſe $\overline{KD}^2 - \overline{CD}^2 = KC^2$; that is, $\frac{1}{4}t^2 - \frac{1}{4}c^2 = \frac{1}{4}b^2$, or $t^2 - c^2 = b^2$; but $pt = c^2$; therefore $t^2 - pt = b^2$, and $t : b :: b : t - p$; or, AB : KH :: KH : AB — LR. Q. E D.

PROPOSITION VIII.

As the Square of any Ordinate, is to the Rectangle of the Abſciſſas, ſo is the Square of the Conjugate, to the Square of the Conjugate added to the Square of the Diſtance of the Foci; that is, $\overline{FG}^2 : AG \times GB :: \overline{ED}^2 : \overline{ED}^2 + \overline{KH}^2$.

DEMONSTRATION.

Becauſe $\overline{KE}^2 = \overline{KC}^2 + \overline{CE}^2$; that is, $\frac{1}{4}t^2 = \frac{1}{4}b^2 + \frac{1}{4}c^2$; or $t^2 = b^2 + c^2$; but (by Prop. 1.) $y^2 : \overline{t - x} \times x :: c^2 : t^2 = b^2 + c^2$; or, $\overline{FG}^2 : AG \times GB :: \overline{ED}^2 : \overline{ED}^2 + \overline{KH}^2$. Q. E. D.

PROPOSITION IX.

As the Square of any Ordinate, is to the Rectangle of its Abſciſſa into the Parameter, ſo is the Differenee between the Square of the conjugate Axe, and the Rectangle of the Abſciſſa into the Parameter, to the Square of the conjugate Axe; that is, $\overline{FG}^2 : BG \times LR :: ED^2 - BG \times LR : \overline{ED}^2$.

DE-

DEMONSTRATION.

From the Equation of the Curve, $t^2 y^2 = c^2 tx - c^2 x^2$; but $t = \frac{c^2}{p}$; therefore, by Substitution, $\frac{c^4 y^2}{p^2} = \frac{c^4 x}{p} - c^2 x^2$, and $c^2 y^2 = c^2 px - p^2 x^2$; that is, $y^2 : px :: c^2 - px : c^2$; or, $\overline{FG}^2 : BG \times LR :: \overline{ED}^2 - BG \times LR : \overline{ED}^2$. Q. E. D.

PROPOSITION X.

As the Square of the conjugate Axe, is to the Square of the transverse Axe, so is the Rectangle of any two Abscissas of the conjugate Axe, to the Square of the Ordinate which divides them; that is, $\overline{DE}^2 : \overline{AB}^2 :: Db \times Eb : \overline{Fb}^2$.

DEMONSTRATION.

Let $Eb = x$, and $Fb = y$; then (by Prop. 1.) $\overline{AB}^2 : \overline{ED}^2 :: AG \times GB : \overline{FG}^2$; but (by *Eu.* 5. 2.) $AG \times GB = \overline{CB}^2 - \overline{Fb}^2$, and $\overline{Cb}^2 = \overline{FG}^2 = \overline{CE}^2 - Db \times Eb$; therefore, by Substitution, $\overline{AB}^2 : \overline{ED}^2 :: \overline{BC}^2 - \overline{Fb}^2 : \overline{CE}^2 - Db \times Eb$; that is, $t^2 : c^2 :: \frac{1}{4} t^2 - y^2 : \frac{1}{4} c^2 - cx + x^2$; which reduced to an Equation, produces $c^2 y^2 = t^2 cx - t^2 x^2$; that is, $c^2 : t^2 :: \overline{c - x} \times x : y^2$; or, $\overline{DE}^2 : \overline{AB}^2 :: Db \times Eb : \overline{Fb}^2$. Q. E. D.

DE-

DEFINITION.

A third Proportional to the Conjugate and transverse Axe, is a Parameter to the conjugate Axe; that is, p being put for the Parameter, $c : t :: t : p$; therefore $cp = t^2$.

PROPOSITION XI.

As the conjugate Axe, is to its Parameter, so is the Rectangle of any two Abscissas of the conjugate Axe, to the Square of the Ordinate which divides them.

DEMONSTRATION.

For t^2, in the last Equation, put its Equal cp; then $cy^2 = cpx - px^2$; that is, $c : p :: \overline{c - x} \times x : y^2$. Q. E. D.

PROPOSITION XII.

As the Square of any Ordinate of the Conjugate, is to the Rectangle of the Abscissas which it divides, so is the Sum of the Squares of the Distance of the Foci, and the conjugate Axe, to the Square of the conjugate Axe.

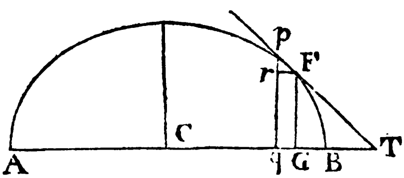
DEMONSTRATION.

By the tenth Prop. $y^2 : cx - x^2 :: t^2 : c^2$, and (by *Eu.* 47. 1.) $t^2 = b^2 + c^2$; therefore, by Substitution, $y^2 : cx - x^2 :: b^2 + c^2 : c^2$; that is, $\overline{Fb}^2 : Db \times Eb :: \overline{KH}^2 + \overline{ED}^2 : \overline{ED}^2$. Q. E. D.

PRO-

PROPOSITION XIII.

In any Tangent to the Ellipse, if, from the Point of Contact, an Ordinate be drawn to the Axe, and the Tangent continued meet the Axe produced, then it will be, as the Distance (in the Axe) between the Center and the Ordinate, is to the Abscissa of that Ordinate, so is the Remainder of the Axe, to (the Distance between the Ordinate and the Intersection of the Tangent with the Axe; that is) the Subtangent, *viz.* CG : GB :: AG : GT.



DEMONSTRATION.

Let FP, an indefinitely small Part of the Curve, be continued to meet the Axe produced in T; draw the Ordinate FG, and parallel to it pq; draw also Fr parallel to the Axe, and for Fr put n, pr, m, and BT, a; then is B$q = x + n$, A$q = t - x - n$, $pq = y + m$, and GT $= a + x$; but, by similar Triangles, pr : Fr :: FG : GT; that is, $m : n :: y : a + x$; therefore $n \times \frac{y}{m} = a + x$, and (by Prop. 2.) $t : p :: tx - x^2 + tn - 2nx - n^2 : y^2 + 2my + m^2$; also, $t : p :: tx - x^2 : y^2$; therefore $ptx -$

px^2

$px^2+ptn-2pnx=ty^2+2tmy$, and $ty^2=ptx-px^2$; consequently $ptx-px^2+ptn-2pnx-2tmy=ty^2=ptx-x^2$; or, $ptn-2npx=2tmy$, and $n=\frac{2tmy}{pt-2px}$; therefore $x+a=\frac{nxy}{m}=\frac{2tmy}{pt-2px}\times\frac{y}{m}=\frac{ty^2}{p}\times\frac{2}{t-2x}=$ (because by the second. Prop. $tx-x^2=\frac{ty^2}{p}$) $\frac{2tx-2x^2}{t-2x}=\frac{tx-x^2}{\frac{1}{2}t-x}$; therefore $\frac{1}{2}t-x:x::t-x:x+a$; or, CG : GB :: AG : GT. Q. E. D.

PROPOSITION XIV.

As the Distance from the Center to the Ordinate drawn from the Point of Contact, is to half the transverse Axe, so is half the transverse Axe, to the Distance from the Center to the Concurring of the Tangent with the Axe produced; that is, CG : CB :: CB : CT.

DEMONSTRATION.

Because, $CT=CG+GT$, and $CT=\frac{1}{2}t+a$, $CG=\frac{1}{2}t-x$; whence (by Prop. 13.) $GT=\frac{tx-x^2}{\frac{1}{2}t-x}$; therefore $\frac{1}{2}t+a=\frac{1}{2}t-x+\frac{tx-x^2}{\frac{1}{2}t-x}=\frac{\frac{1}{4}t^2}{\frac{1}{2}t-x}$; that is, $\frac{1}{2}t-x:\frac{1}{2}t::\frac{1}{2}t:\frac{1}{2}t+a$; or, CG : CB :: CB : CT. Q. E. D.

PRO-

PROPOSITION XV.

As the Diſtance from the Center to the Ordinate drawn from the Point of Contact, is to half the Tranſverſe, ſo is the Abſciſſa of that Ordinate, to the external Part of the Tranſverſe; that is, CG : CB :: GB : BT.

DEMONSTRATION.

By Prop. 14. $\frac{1}{2}t + a = \frac{\frac{1}{4}t^2}{\frac{1}{2}t - x}$; therefore $\frac{1}{4}t^2 + \frac{1}{2}ta = \frac{1}{2}tx - ax = \frac{1}{4}t^2$, and $a = \frac{\frac{1}{2}tx}{\frac{1}{2}t - x}$; that is, $\frac{1}{2}t - x : \frac{1}{2}t :: x : a$; or, CG : CB :: GB : BT. Q. E. D.

PROPOSITION XVI.

As the Diſtance from the Center to the Ordinate drawn from Point of the Contact, is to half the tranſverſe Axe, ſo is the greater Abſciſſa of that Ordinate, to the tranſverſe Axe added to the external Part; that is, CG : CB :: AG : AT.

DEMONSTRATION.

By the 15, $a = \frac{\frac{1}{2}tx}{\frac{1}{2}t - x}$; therefore $t + a = t + \frac{\frac{1}{2}tx}{\frac{1}{2}t - x} = \frac{\frac{1}{2}t^2 - \frac{1}{2}tx}{\frac{1}{2}t - x}$; that is, $\frac{1}{2}t - x : \frac{1}{2}t :: t - x : t + a$; or, CG : CB :: AG : AT. Q. E. D.

PROPOSITION XVII.

As the greater Abſciſſa of the Ordinate drawn from the Point of Contact, is to the Sum of the Transverſe and external Part, ſo is the leſs Abſciſſa of that Ordinate, to the external Part; that is, AG : AT :: BG : BT.

DEMONSTRATION.

By the 15, $\frac{1}{2}t - x : \frac{1}{2}t :: x : a$, and by the 16, $\frac{1}{2}t - x : \frac{1}{2}t :: t - x : t + a$; therefore, by Equality, $t - x : t + a :: x : a$; or, AG : AT :: BG : BT.

PROPOSITION XVIII.

As the Diſtance from the Center to the Concurring of the Tangent, is to half the Tranſverſe, ſo is the external Part, to the Abſciſſa of the Ordinate drawn from the Point of Contact; that is, CT : CB :: BT : BG.

DEMONSTRATION.

By the 15, $\frac{1}{2}ta = \frac{1}{2}tx + xa$; therefore $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t + a}$, and $\frac{1}{2}t + a : \frac{1}{2}t : a : x$; or, CT : CB :: BT : BG.

PROPOSITION XIX.

As half the Tranſverſe added to the external Part, is to the Tranſverſe added to the external Part,

Part, ſo is the external Part, to the Subtangent; that is, CT : AT :: BT : GT.

DEMONSTRATION.

By the 18, $x = \frac{\frac{1}{2}ta}{\frac{1}{2}t + a}$; therefore $x + a = a + \frac{\frac{1}{2}ta}{\frac{1}{2}t + a} = \frac{ta + a^2}{\frac{1}{2}t + a}$, and $\frac{1}{2}t + a : t + a :: a : x + a$; or, CT : AT :: BT : GT. Q. E. D.

PROPOSITION XX.

As the greater Abſciſſa of the Ordinate drawn from the Point of Contact, is to half the Tranſverſe, ſo is the Subtangent, to the external Part; that is, AG : CB :: GT : BT.

DEMONSTRATION.

By the 15, $\frac{1}{2}t - x = \frac{\frac{1}{2}tx}{a}$; therefore $t - x = \frac{1}{2}t + \frac{\frac{1}{2}tx}{a} = \frac{\frac{1}{2}ta + \frac{1}{2}tx}{a}$, and $t - x : \frac{1}{2}t :: x + a : a$; or, AG : CB :: GT : BT. Q. E. D.

PROPOSITION XXI.

As the Tranſverſe added to the external Part, is to half the Tranſverſe, ſo is the Subtangent, to the Abſciſſa; that is, AT : CB :: GT : GB.

DEMONSTRATION.

By the 18, $\frac{1}{2}t+a=\frac{\frac{1}{2}ta}{x}$; therefore $t+a=\frac{1}{2}t+\frac{\frac{1}{2}ta}{x}=\frac{\frac{1}{2}tx+\frac{1}{2}ta}{x}$, and $t+a:\frac{1}{2}t::x+a:x$; or, AT : CB :: GT : GB. Q. E. D.

PROPOSITION XXII.

The Ordinate drawn from the Point of Contact, divided by the Subtangent, is equal to the Quotient of the Distance between the Center and that Ordinate divided by that Ordinate, multiplied by the Parameter divided by the transverse Axe; that is, $\frac{GF}{GT}=\frac{CG}{GF}\times\frac{p}{t}$.

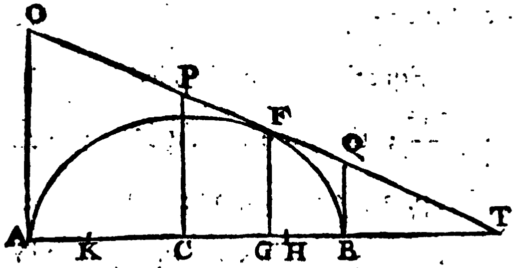
DEMONSTRATION.

By the 13, $tx-x^2=\overline{\frac{1}{2}t-x}\times\overline{x+a}$; and (by Prop. 2.) $t:p::tx-x^2:y^2::\overline{\frac{1}{2}t-x}\times\overline{x+a}:y^2$; therefore $ty^2=p\times\overline{\frac{1}{2}t-x}\times\overline{x+a}$; which being divided by $\overline{x+a}\times ty$, it produces $\frac{y}{x+a}=\frac{\overline{\frac{1}{2}t-x}}{y}\times\frac{p}{t}$. Q. E. D.

PROPOSITION XXIII.

If Perpendiculars be drawn from the Extremities of the Transverse, and from the Center, meeting any Tangent, and also if from the Point of Con-

Contact, be drawn an Ordinate, these four Lines will be proportional; that is, AO : CP :: FG : BQ.



DEMONSTRATION.

By the 19, TA : TC :: TG : TB; therefore (by *Eu.* 4. 6.) AO : CP :: FG : BQ. Q. E. D.

COROLLARY.

AO × BQ = CP × FG.

PROPOSITION XXIV.

If Perpendiculars be drawn from the Extremities of the Transverse, meeting any Tangent; then the Rectangle of these Perpendiculars, will be equal to the Rectangle of the greatest and least Distance of either of the Foci from the Vertices; that is, AO × BQ = AH × BH = BK × AK.

DEMONSTRATION.

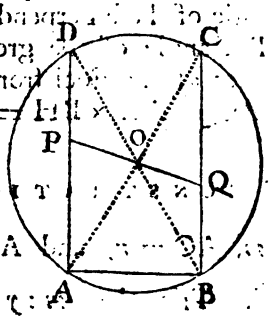
Let BQ $= m$, AO $= n$, and AK = BH $= q$; then, by similar Triangles, $m : y :: a : a + x ::$

(by

(by Prop. 20.) $\frac{1}{2}t : t - x$; and $n : y :: t + a$ $a + x ::$ (by Prop. 21.) $\frac{1}{2}t : x$; therefore $m = \frac{\frac{1}{2}ty}{t-x}$ and $n = \frac{\frac{1}{2}ty}{x}$; hence, if the respective Sides of the two Equations be multiplied by each other, $mn = \frac{\frac{1}{4}t^2y^2}{tx-x^2}$; therefore $mn : \frac{1}{4}t^2 :: y^2 : tx - x^2$ :: (by Prop. 2.) $p : t :: \frac{1}{4}pt : \frac{1}{4}t^2$; but $mn = \frac{1}{4}pt =$ (by Prop. 3.) $\overline{t-q} \times q$; therefore AO × BQ = AH × BH = BK × AK. Q. E. D.

LEMMA.

If on the Extremities of any Subtense of a Circle, as AB, the Perpendiculars AD, BC be erected meeting the Periphery in the Points D, C; and from these Points to the opposite Extremities, B and A, of that Subtense, be drawn two right Lines DB and CA, they will intersect each other in (O) the Circle's Center; through which, also, if a right Line be drawn any how, it will make the alternate Segments of the Perpendiculars equal.

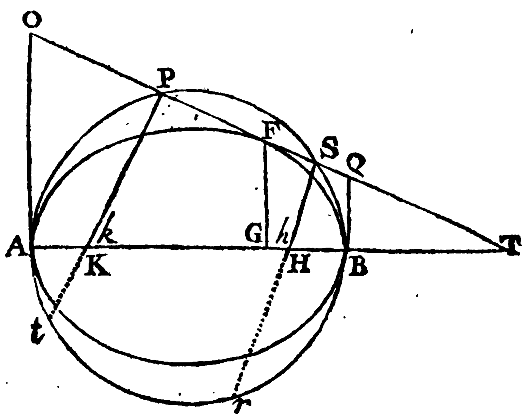


DEMONSTRATION.

By Hypothesis the Angles B and A are right; therefore (by *Eu.* 31. 3). BD and AC are Diameters, and consequently the Point of Intersection the Center of the Circle; but the Triangles OPD, OQB are similar; therefore BO : BQ :: DO : DP, and consequently, since BO = DO, BQ = DP. Q. E. D.

PROPOSITION XXV.

If from the Intersections (P, S) of a Circle, whose Diameter is the transverse Axe, with any Tangent, Perpendiculars P*k*, S*h* be drawn, they will cut the transverse Axe in the focal Points; that is, the Points *k*, *h* coincide with K, H.



DEMONSTRATION.

The Triangles TBQ, ATO are similar to the Triangles TS*h*, TP*k*, each having a right Angle, and

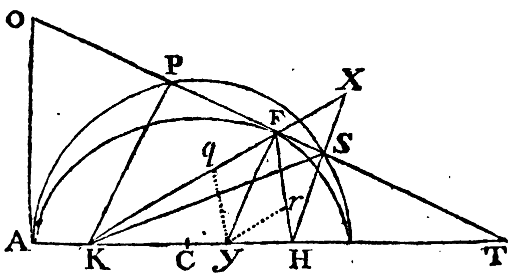
and the Angle T common; therefore AO : Pk :: Sb : BQ, and AO × BQ = Pk × Sb = (by the precedent *Lemma*) Pk × kt; or, br × Sb = (by *Eu.* 35. 3.) Ak × Bk, or Bb × Ab; but AO × BQ = Ak × BK or AH × HB (by Prop. 24.); therefore the Points H, b and K, k are coincident. Q. E. D.

COROLLARY.

It is manifest that, KP × HS = $\frac{1}{4}pt$; because KP × HS = AK × BK = $t - q \times q$ = (by Prop. 3.) $\frac{1}{4}pt$.

PROPOSITION XXVI.

If to any Point of the Curve right Lines be drawn from the Foci, and one of the Lines be continued; then a right Line bisecting the external Angle, will touch the Curve in the angular Point.



DEMONSTRATION.

Take FX = FH; then (because by Hypothesis, the Angle TFX = HFT) if you take any Point S, in the Line FT; HS = SX (by *Eu.* 4. 1.). Draw KS; then KS + SX = KS + HS is greater than KX; or its Equal AB, and therefore the Point S, is without the Curve; for if it were in the Curve KS + HS (by the Genesis) = AB.

PRO-

PROPOSITION XXVII.

Lines drawn from the Foci to the Point of Contact, make equal Angles with the Tangent.

DEMONSTRATION.

By the precedent Prop. the Angle HFT = TFX = (by *Eu.* 15. 1.) KFO. Q. E. D.

PROPOSITION XXVIII.

A right Line perpendicular to the Tangent at the Point of Contact, bisects the Angle formed by Lines drawn from the Foci to the same Point; that is, if FY be perpendicular to OT; then the Angle KFY = HFY.

DEMONSTRATION.

The Angle PFY = TFY, by Hypothesis, from which if there be taken the Angle KFP = HFS (by Prop. 27.) there will remain the Angle KFY = HFY. Q. E. D.

PROPOSITION XXIX.

If, on the Tangent, at the Point of Contact, a Perpendicular be drawn meeting the Axe, it will divide the Distance between the Foci, in the same Proportion, as Lines drawn from the Foci to the same Point; that is, FK : FH :: KY : HY.

DEMONSTRATION.

In the Triangle HFK, the Angle KFY = HFY (by Prop. 28.); therefore (by *Eu.* 3. 6.) FK : FH :: KY : HY. Q. E. D.

PROPOSITION XXX.

If, on the Tangent, at the Point of Contact, a Perpendicular be drawn, and if, from the Point where that Perpendicular meets the Axe, Lines be drawn perpendicular, to Lines drawn from the Foci to the Point of Contact; then the Distance on these Lines, from the Point of Contact, to the Perpendiculars, will be equal to half the Parameter of the Axe; that is, $Fq = Fr = \frac{1}{2}p$.

DEMONSTRATION.

From the Points S, P, where a Circle on the Transverse cuts the Tangent, draw the Lines SH, PK to the Foci, which will be perpendicular to PT, by 25. and consequently parallel to FY; continue KF, HS, till they concur in X; then $KX = t$, and $HX = 2HS$ by the 26. and because the Triangles KFY, KPF are respectively similar to the Triangles KXH, qFY; therefore $KX : HX :: FK : FY :: KP : Fq$, and $KX \times Fq = HX \times KP$; but $\frac{1}{2} HK \times Fq = \frac{1}{2} HX \times KP$; that is, $\frac{1}{2} t \times Fq = HS \times KP =$ (by Prop. 25.) $\frac{1}{4} pt$; therefore $Fq = \frac{1}{2} p$; but (by Prop. 28.) the Angle $YFq = YFr$, and (by *Eu.* 26. 1.) $Fr = Fq$; therefore $Fq = Fr = \frac{1}{2} p$. Q. E. D.

To be continued.

A N-

James Lee

ANSWERS

TO THE

PROBLEMS

JL PROPOSED IN Lee

NUMBER I.

PROBLEM I. *Answered by Mr.* Thomas Hulme *of* London.

LET x represent the Number of Persons, that agreed with the Waterman; then $\frac{x}{4}+3$ will be the Number of Persons taken in by the Way, and $\frac{3x}{2}+18$ the Pence they paid; half of which divided by x, will be $\frac{3}{4}+\frac{9}{x}$ what each of the first Persons gained, by Means of those taken in afterwards, which by the Conditions of the Problem $=1$; therefore $x=\frac{9}{1-\frac{3}{4}}=36$.

PROBLEM II. *Answered.* R.

DEFINITION.

The Weight of a Body compared with that of another Body of equal Magnitude, is called its specific Gravity.

Now the specific Gravities of Bodies may be thus determined by Experiment: Let the Body whose specific Gravity is required, be first weighed in Air, afterwards in Water; then the specific Gravity of the Body, is to that of Water, as the Weight of the Body in Air, to the Weight lost in Water: For *(by the Definition)* the Weight of Water of the same Magnitude with the Body, is to the Weight of the Body, as the specific Gravity of Water, to the specific Gravity of the Body: Hence it follows, if the Water be specifically heavier than the Body; that the specific Gravity of the Body, is to that of Water, as the immerged Part of the Body, to the Magnitude of the Body: But the specific Gravity is as the Density; therefore it may very justly be as Mr. *Turner* expresses it: As the Density of Water, to the Density of the Body, so is the Magnitude of the Body, to the Magnitude of the Space which the Body possesses in the Water, and so is one Foot, the Side of the Cube, to the Part thereof immerged: Therefore let a ($= 1728 \times 0.527458$, Ounces-Troy) be the Weight of a Cubic-Foot of Water, and b ($= 1728 \times 0.489008$ such Ounces) that of Oak; then $a - p$ ($= 66.4416 \times \frac{1}{12} = 5.5368$ Pounds) the Weight to be laid on, and $a : b :: 1 : \frac{a}{b}$, the Part thereof immerged; consequently $1 - \frac{a}{b}$ ($= 1 - \frac{0.489008}{0.527458}$, or $12 - \frac{0.489008 \times 12}{0.527458}$ Inches) the Thickness of the Part above the Water.

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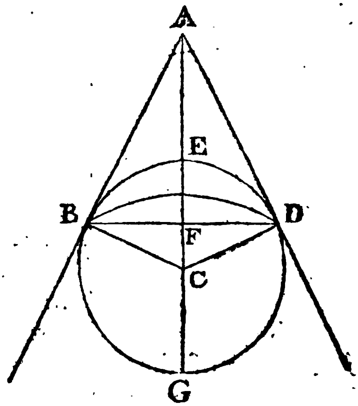
PROBLEM III. *Answered by Mr.* John Turner *of* London.

Since the Curve-Surface of any Frustum of a Sphere, is to the whole Surface, as the Height of the Frustum to the Diameter of the Sphere; EF will be to EG as 2 to 5; therefore EF will be to EC as 2 to $2\frac{1}{2}$, or as 4 to 5, and FC to EC as 1 to 5; but by similar Triangles, CE : BC :: BC : AC; that is, 1 : 5 :: B : 5 BC = AC; therefore AE = 4 BC, or twice the Diameter.



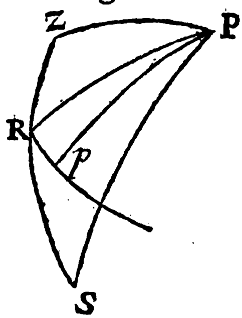
The same answered. R.

If GBED, the Section of the Earth, with the Plane of the Meridian be a Circle, and Lines be drawn to the same according to the Import of the Problem, as in the Figure: It will appear that, $GF \times FE = CF \times FA$, or by the Property of the Sphere, mentioned in the foregoing Solution, $\frac{3GE \times 2.GE}{25} = \frac{GE \times FA}{10}$; therefore $FA = \frac{12\,GE}{5}$, and consequently $AE = 2\,GE$, the same as above.

PRO-

PROBLEM IV. *Answered by Mr.* John Turner *of* London.

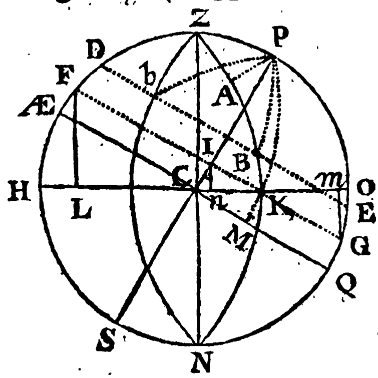
Let ZP be the Complement of Latitude, and the Angle Z the Azimuth of the Plane; also, let S represent the Place of *Sirius* when in the Plane, *p* that of *Pollux* at the same Time, and R*p* a Part of the Parallel of Declination described by *Pollux* in (56′ 20″) the given Time: Then, since the Angle RP*p* answering to the given Time, and *p*PS that answering to the Difference of Right-Ascensions of the two Stars are given, their Sum RPS will likewise be given; therefore in the Triangle SPR will be given two Sides and the Angle included, whence the Angle S will be known; then in the Triangle ZPS will be given two Sides and an Angle, from whence the Angle Z will also be known, whose Difference from a right Angle is the Declination required; which (supposing the Right-Ascension of *Pollux* = 112° 12′, the Declination North 28° 51′, the Right-Ascension of *Sirius* 98° 29′, the Declination South = 16° 17′) will be = 54° 44′.



PROBLEM V. *Answered.* R.

Let HZON be the Meridian, HO the Horizon, ÆQ the Equator, making an Angle with the same equal to the Complement of Latitude, NZ the prime Vertical, PS the Axis of the Sphere and Hour Circle of Six, in the required Latitude OP, NKZ an Azimuth Circle, upon the Plane of which

which the erect declining Dial is supposed to be drawn; also let PA be perpendicular to that Azimuth Circle: Then in the right-angled spherical Triangle AZP, it is manifest, that PZ may represent the Complement of the Latitude, AP the Height of the Style, AZ the Distance of the Substyle from the Meridian, the Angle AZP the Complement of the Plane's Declination, and the Angle APZ the Inclination of the Meridians: Therefore, since AP, AZ and the Angle AZC, are equal to each other, by the State of the Problem, let the Sine of any one of them be



put $= x$; then the Tangent of AZ $= \frac{x}{\sqrt{1-x^2}}$, and that of the Angle AZP $= \frac{\sqrt{1-x^2}}{x}$; whence (*by spherical Trigonom.*) $x : 1 :: \frac{x}{\sqrt{1-x^2}} : \frac{\sqrt{1-x^2}}{x}$; therefore $\sqrt{1-x^2} = \frac{x}{\sqrt{1-x^2}}$, and consequently $x^2 + x = 1$, or $x = \frac{\sqrt{5}-1}{2} = 0.61803$, &c. the Sine of (38° 10′) the Plane's Declination; whence the Latitude = 38° 10′.

But to find what Time the Sun comes on and goes off the Plane: Let EB*b*D be the Parallel of Declination described by the Sun on the Day proposed; then it is manifest, that the Sun first comes on the Plane at B, where it is coincident with the Azi-

Azimuth Circle, already defined; therefore, if through B and P the Pole a great Circle be described, in the spherical Triangle BZP will be given two Sides and an Angle opposite to one of them; whence the Angle BPZ, shewing the Time before Noon that the Sun comes on the Plane, will also be given: Moreover, if N*b*Z be the farther Part of the Azimuth Circle, it is also manifest, that the Sun goes off the Plane at *b*, where it is coincident with that Part; therefore, if through *b* and P the Pole, a great Circle be described, in the spherical Triangle *b*ZP will be given two Sides and an Angle opposite to one of them, as in the former Case; whence the Angle *b*PZ, shewing the Time past Noon that the Sun goes off the Plane, will also be given.

But if it were also required, to find the Declination described by the Sun, when it continues the longest on the Plane: Through K, the Intersection of the Horizontal and Azimuth Circle let there be drawn FG, and also through K and P the Pole, let a great Circle be described, meeting the Equator in M; then in the right-angled spherical Triangle KCM will be given CK, the Plane's Declination, and the Angle KCM the Complement of Latitude; whence KM, the Sun's Declination, will likewise be given.

PROBLEM VI. *Answered by Mr.* John Turner *of* London.

Let HPOS, in the former Figure, be an Orthographick Projection of the Sphere; in which HO represent the Horizon, ÆQ the Equator, FG the Parallel of Declination, PS the Axis of the Sphere; also, let I*n* represent the Sun's Altitude at Six, and let the Perpendiculars FL and G*m* be drawn: Then, by the Property of the

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Ellipsis, and similar Triangles, it will be, as CM : ÆM :: IK : FK :: In : FL; and CM : MQ :: IK : KG :: In : mG; whence the Arches FH ($50^{\circ}\ 45'\ 44''$) and GO ($14^{\circ}\ 52'\ 2''$) become known: Thus far Mr. *Turner*: But in order to obtain the Declination and Latitude from thence, it is manifest that the Arch HF − ÆF − GO = ÆF; therefore $ÆF = \frac{HF - GO}{2}$, and consequently $HÆ = \frac{HF - GO}{2} + GO = \frac{HF + GO}{2}$: Hence proceed the two following Theorems, generally made use of, for that Purpose.

THEOREM I.

From the Meridian Altitude, substract the Depression at Midnight, and the Half of that Remainder, will be equal to the Declination.

THEOREM II.

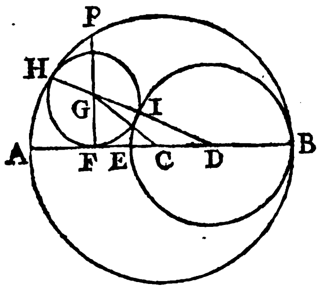
To the Meridian Altitude, add the Depression at Midnight, and the Half of that Sum, will be equal to the Complement of the Latitude.

Whence the Latitude, in the present Case, is $56^{\circ}\ 41'\ 7''$ and the Declination $18^{\circ}\ 26'\ 51''$; which answers to the second Day of *May*. R.

PROBLEM VII. *Answered by Mr.* John Turner *of* London.

Join the Centers of the given Circles, and on AB let fall the Perpendicular GF; putting $BD = a$, $BC = b$, $BF = x$, and $FG = y$; then (by 47. *Eu.* 1.)

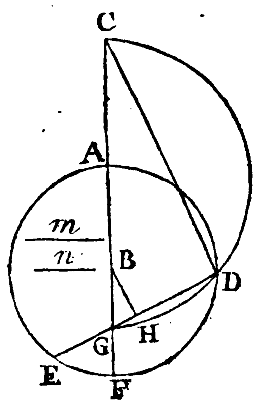
we shall have $\overline{x-a}|^2+y^2=\overline{a+y}|^2$ and $\overline{x-b}|^2+y^2=\overline{b-y}|^2$:



From the former of which $y=\frac{x^2-2ax}{2a}$, and from the latter $y=\frac{2bx-x^2}{2b}$; therefore $\frac{x^2-2ax}{2a}=\frac{2bx-x^2}{2b}$, and consequently $x=\frac{4ab}{a+b}$: Whence this Construction: Take BF, a fourth Proportional to BD+BC, BE and BA, and make FP perpendicular to BA; in which take FG a fourth Proportional to BE, BF and FE; then will G be the Center, and FG the Radius of the Circle to be described.

PROBLEM VIII. *Answered by the same.*

Let AEFD be the given Circle; B the Center thereof, and G the given Point; and let the Lines m, n be in the given Ratio of the Parts DG, EG: Take BC to BG as $m+n$ to $m-n$, and upon the Diameter GC let a Circle be described, cutting the former in D; draw DGE and the Thing is done.



DEMONSTRATION.

Draw CD, and BH parallel thereto; then $m+n : m-n$, or $\frac{m+n}{2} : \frac{m-n}{2} :: CB : BG :: DH : GH$; therefore by Compoſition and Diviſion $m : n :: DG : EG$. Q. E. D.

Otherwiſe Algebraically. *R.*

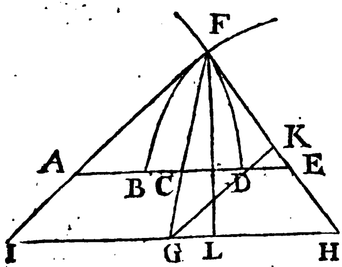
Let $GF = a$, $AG = b$, the given Ratio of the Parts as m to n, and the leſſer of them $= x$; then the greater will be $\frac{mx}{n}$; whence, by the Property of the Circle, $\frac{mx^2}{n} = ab$, and conſequently $x = \sqrt{\frac{nab}{m}}$: From whence proceeds the following Conſtruction, given by *Tycho Oxonienſis*. Let the given Point be G, and the given Ratio of the Parts as R to S: Then take the Line $m : FG :: S : R$, and between AG and m, find a mean Proportional EG, which apply in the given Circle from G to the Periphery; continue EG till it cuts the Circle in D, and DG will be to $EG :: R : S$. For (by Conſtruction) $AG : EG :: EG : m$, and $AG : EG :: DG : FG$ (by 35. *Eu.* 3.); therefore $DG : FG :: EG : m$, and, by Permutation, $DG : EG :: FG : m$; that is, as R to S. Q. E. D.

PROBLEM IX. *Anſwered by Mr.* John Turner.

CONSTRUCTION.

Draw AE at Pleaſure, which biſect at C, and take AB to BC in the Ratio of one Side to the bi-

fecting Line, and CD to DE in the Ratio of the bifecting Line to the other Side, then according to the Method laid down in Lemma Page 310 of Mr. *Simpson*'s Algebra, let two Circles be defcribed cutting each other in F; draw FA, FE and FC, in which produced, if need be, take FG equal to the bifecting Line, and draw IGH parallel to AE, interfecting FA and FE produced in I and H; then FIH will be the Triangle that was to be conftructed. The Demonftration of which is manifeft from the before mentioned Lemma.



The same anfwered. N.

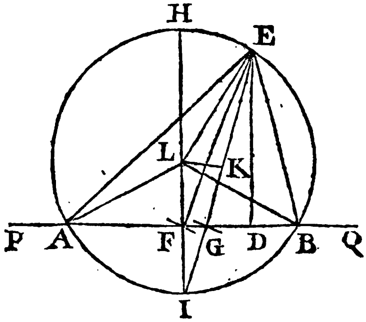
Let FG be the given Line bifecting the Bafe, FI and FH the two given Sides of the Triangle: Then if GK be drawn parallel to FI, it will manifeftly bifect FH, and be equal to the Half of FI; therefore in the Triangle FGK there will be given all the Sides, from whence the Angle GKH will be given, as well as the Angle GKF; then in the Triangle GKH will be given two Sides, and an Angle included; whence GH will be given, and confequently the Bafe of the Triangle.

PROBLEM X.. *Anfwered by Mr.* John Turner.

CONSTRUCTION.

From any Point D, in the indefinite Line PQ, draw DE perpendicular to PQ and equal to the given

given Perpendicular; then upon E with Radii respectively equal to the two given bisecting Lines, describe two Arcs cutting PQ in F and G; draw HFI perpendicular to PQ, also draw EG which produce to meet HI in I; bisect EI with the Perpendicular KL, meeting HI in L, then upon L with LI as Radius, describe the Circle AHBI cutting PQ in A and B; join A, E and B, E, then ABE will be the Triangle that was to be constructed.



DEMONSTRATION.

Join E, F; L, E; L, B and L, A: Because LK is perpendicular to and bisects IE, it is evident the Circle passes through the Point E. Moreover, because FL is perpendicular to AB, and LA equal to LB, AF will be equal to FB; wherefore it is evident that FE bisects the Base, and that the Arches IA, IB as well as the Angles AEI, and BEI are equal. Q. E. D.

Method of Calculation.

As the Line bisecting the vertical Angle, is to the Perpendicular, so is Radius, to the Cosine of half the Difference of the Angles at the Base: And as the Line bisecting the Base, is to the Perpendicular, so is Radius, to the Cosine of an Angle, which substract from the Difference of the Angles at the Base found by the preceding Proportion; then say, as the Sine of the said Angle, is to the Sine

Sine of the Remainder, so is Radius, to the Cosine of the vertical Angle; whence all the Angles are given, and consequently the Sides.

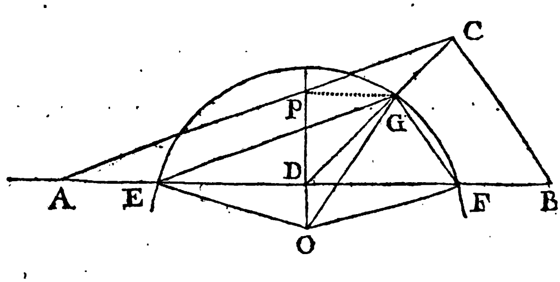
Mr. Thomas Hulme's *Answer to the same.*

Since EF, EG, ED are given; FG and GD, because they both from thence may be found, are said to be given; therefore if the Triangle be inscribed in a Circle, and those Lines be put equal to, a, b, c, d, h respectively; we shall then have, by putting the Semi-Base $= x$; $AG = x + d$ and $GB = x - d$; whence, by similar Triangles, GD : EG :: FG : $GI = \frac{bd}{h}$, and, by the Property of the Circle, $AG \times BG = GI \times EG$; therefore $x^2 - d^2 = \frac{db^2}{h}$, and consequently $x = \sqrt{\frac{db^2}{h} + d^2}$.

PROBLEM XI. *Answered by Mr.* John Turner.

CONSTRUCTION.

Draw AB at Pleasure, in which take $ED = DF$, and upon EF let a Segment of a Circle be described to contain an Angle, equal to the given Angle at the Vertex; make BDG equal to the Angle



which the bisecting Line makes with the Base, and produce DG, if need be, so that DC may be equal

equal to the bisecting Line; join E, G and F, G and draw CA and CB parallel to EG and FG respectively, then ABC will be the Triangle required.

DEMONSTRATION.

Because AC and BC, are parallel to EG and GF; the Angles ACD and BCD will be equal to EGD and FGD respectively; therefore ACD + BCD = EGD + FGD = EGF = the given Angle at the Vertex by Construction: Also, because of the similar Triangles, ACD, EGD and BCD, FGD, we shall have GD : DE (DF) :: GC : AE = FB; therefore AD = DB. Q. E. D.

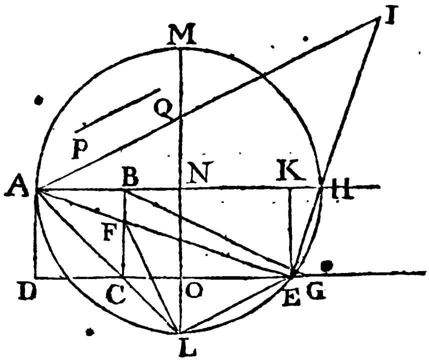
Method of Calculation. *R.*

From the Center O of the described Circle, conceive OE, OG and OF to be drawn, also ODP to be perpendicular, and GP to be parallel to EF: Then if the Value of EF be assumed at pleasure, there will be given in the right-angled Triangle ODF all the Angles, and the Side DF; whence the other two Sides OD, and OF = OG will also be given; and therefore, since the Angle GDO is given; in the Triangle ODG there will be given, the two Sides OD, OG and an Angle opposite to one of them; whence the Angle DOG, the Difference of the Angles at the Base, will likewise be given; but their Sum is given, because the Angle at the Vertex is given; therefore the Angles themselves from hence become known: Wherefore in the Triangle ACB are given, the Angles at the Base, the Angles which the bisecting Line makes with the Base, and the Line itself; whence the Sides will also be given.

PROBLEM XII. *Anſwered by Mr.* John Turner.

CONSTRUCTION.

Make CG equal to the given intercepted Line, and join B, G; in AB produced take $BH = BG$, and upon the Diameter AH let a Circle be deſcribed cutting DG in E, draw AE and the Thing is done.



DEMONSTRATION.

Draw EK and the Diameter ML perpendicular to DG; alſo, draw EHI, ſo, that EI may be equal to EA, and join A, I; A, L, and L, E: Then by the Property of the Circle $LE^2 = LM \times LO$, therefore $2LE^2 = LM \times 2LO$; but $LM = AH$, is $= BG + AB$ by Couſtruction, and $2LO = 2NL - 2NO = ML - 2NO = ML - 2AB = BG - AB$; conſequently $2LE^2 = \overline{BG + AB} \times \overline{BG - AB} =$ (by *Eu.* 2. 5.) $BG^2 - AB^2 = BG^2 - BC^2 = CG^2$: Therefore, becauſe of the ſimilar Triangles ALE, AHI, it will be as $2AL^2 : 2LE^2\ (CG^2) :: AH^2 : HI^2$; but the Antecedents being equal, the Conſequents will likewiſe be equal, that is $CG^2 = HI^2$, or $CG = HI$.

HI. Moreover, because the equiangular Triangles ABF, EHK have the homologous Sides AB and EK equal to each other, the Side AF will also be equal to the Side EH, which being taken from the equal Quantities AE and EI, will leave FE = HI = CG. Q. E. D.

Method of Calculation.

In the Triangle LOE are given the two Sides LO, LE and the right Angle LOE; whence the Side OE will be given, and consequently the Point E in the Side of the given Square produced.

The same answered by Tycho Oxoniensis.

Let CG be the given Line: Take PQ : CG :: DC : AC, and AL : PQ :: PQ : CL; from the Point L apply the Line FL = PQ, and through the Points A and F draw AE; then EF is the Line required.

For since by Construction, AL : PQ :: PQ : CL; that is, AL : FL :: FL : CL, the Triangles ALF and CLF, will be equiangular, and the Angle AFL = FCL = ACE; but the Angle LAF is common to both the Triangles AFL, ACE; therefore AC : CE :: AF : FL, that is to PQ; and by Reason of the Parallels AD, and CF; DC : CE :: AF : FE; therefore by Equality DC : AC :: PQ : FE; but, by Construction, DC : AC :: PQ : CG; therefore CG = FE. Q. E. D.

PROBLEM XIII. *Answered by* Tycho Oxoniensis.

CONSTRUCTION.

At the Point A, the Extremity of the Line AB, joining the two given Points, make the Angle BA*m* equal to the given Angle D; also make AO

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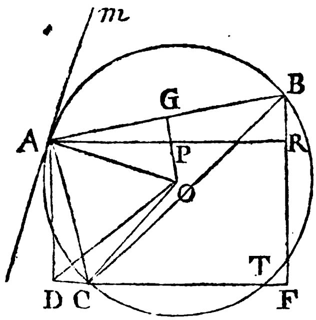
perpendicular to A*m*, and upon G the Middle of AB erect another Perpendicular, meeting the former in O; on the Point O as a Center, with the Radius AO, describe the Circle ABCT, cutting the given right Line DF in C; join AC, BC and these Lines shall comprehend the given Angle D. The Demonstration of which is very obvious, from the 20 and 32. *Eu.* 3.



Method of Calculation.

Draw AR parallel, and AD, BF perpendicular to DF; then in the Triangle BAR will be given AB, BR (= BF — AD) and the right Angle ARB; whence the other two Angles ABR, BAR and the Side AR will be given; but the Angle GAO is given, (from the Construction) and AG = $\frac{AB}{2}$; therefore the Radius AO, and consequently the Angle PAO will be given; the Difference between which Angle and a right one, is equal to the Angle DAO; therefore in the Triangle DAO are given two Sides and an Angle included, whence OD, the Angle ADO and consequently the Angle ODC will likewise be given; then in the Triangle ODC will be given the two Sides OD, OC (= AO) and the Angle ODC, whence DC will be given, and consequently the Lines AC, CB containing the given Angle.

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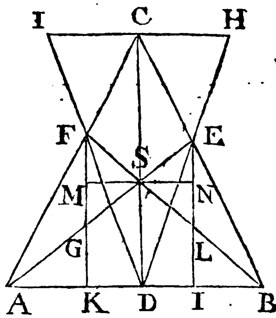
PROBLEM XIV. *Answered by Mr.* John Turner.

Draw ICH parallel to AB, intersecting DE and DF produced in H and I: Then since, by similar Triangles, BE : BD :: CE : CH $= \frac{BD \times CE}{BE}$, and AF : AD :: CF : CI $= \frac{AD \times CF}{AF}$; we shall have CH : CI :: $\frac{BD \times CE}{BE}$: $\frac{AD \times CF}{AF}$ :: BD × CE × AF : AD × CF × BE. But the two last Terms are equal by a known Property of the Triangle; therefore CH and CI are also equal, and consequently the Angle IDC = the Angle HDC. Q. E. D.



The same answered by the Proposer. *R.*

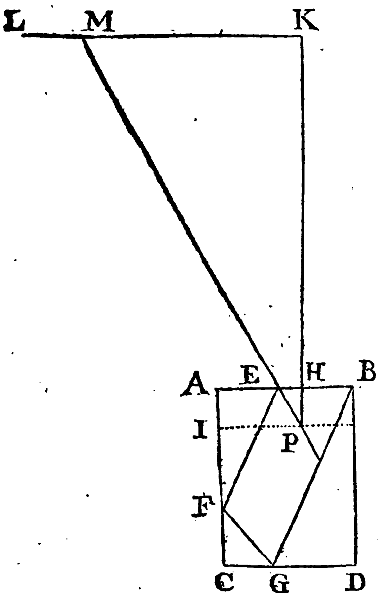
To the Base AB, from the Points F and E, let fall the Perpendicular FK and EI, and parallel to the same through S, the common Intersection of the Lines AE, BF and CD, draw MN: Then, by similar Triangles, AC : AF :: CS : GF :: CD : FK; therefore, by Permutation, CS : CD :: GF : FK; after the very same Manner, it will appear that CS : CD :: EL : EI; whence, by Equality, GF : FK :: EL : EI, or again, by Permutation, GF : EL :: FK : EI: But GF : EL :: FS : LS :: MS : NS; whence again, by Equality, FK : EI :: MS : NS, or as KD : DI; therefore the two Triangles DFK, DEI, having one Angle K equal to one Angle I, and the Sides about the other An-

 gles.

gles DFK, DEI proportional, are equiangular (*Eu.* 7. 6.); and consequently the Angle FDK, equal to the Angle IDE. Q. E. D.

PROBLEM XV. *Answered by Mr.* John Turner.

Let PEFGB be the Path of the Ball, and make PH perpendicular to AB, and PI to AC: Then, because the Angles of Incidence and the Angles of Reflection are equal, and the Angles at H, A, C, and D right Angles, the Triangles PHE, FEA, FCG and BGD are all similar; but the Sum of all the Perpendiculars PH + AF + FC + BD is given = PH + 2 BD, and the Sum of the Bases EH + AE + CG + DG is also given = AH + CD; therefore it will be as PH + 2 BD : AH + CD :: PH : HE; whence this

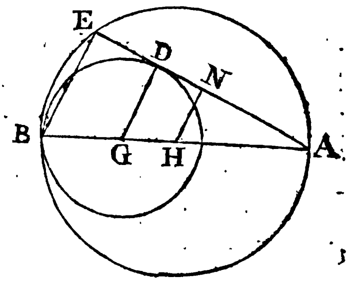


CONSTRUCTION.

In PH produced take HK = 2 BD; and in KL perpendicular to PK, take KM = AH + CD, and draw PM for the Direction required.

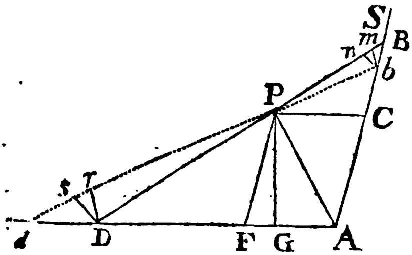
PROBLEM XVI. *Answered by Mr.* Thomas Leigh.

From the Point of Contact D, to the Center of the lesser Circle, draw the Radius DG, and parallel to the same from H the Center of the greater, draw also the right Line HN: Then by putting GH $=a$, ED $=b$, and BH, the Radius of the greater Circle, $=x$; we shall have AG $=a+x$, BG $=$ GD $=x-a$; whence, (by *Euclid* 36. 3.) DA $=2\sqrt{ax}$; but, by similar Triangles, AB : AG :: AE : AD; therefore, by Division, AB — AG : AG :: AE — AD : AD; that is, BG : AG :: ED : AD, or $x-a : a+x :: b : 2\sqrt{ax}$; therefore $\overline{x-a}|^2 \times 4ax = \overline{a+x}|^2 \times b^2$; which Equation, by Reduction, becomes $x^3 - \overline{2a+\frac{b^2}{4a}} \times x^2 + \overline{a^2 - \frac{b^2}{2}} \times x = \frac{b^2 a}{4}$, from whence the Value of x may be determined, and consequently the Diameter of the lesser Circle.



PROBLEM XVII. *Answered by Mr.* Thomas Leigh.

Let A*d* and AS be the two right Lines given by Position, P the given Point, and DPB the Line required: Then if PF be drawn parallel to AB,



AB, and PG perpendicular to Ad, we shall have, by putting $PG = a$, $FG = c$, $AG = n$, and $DG = x$; $DF = x - c$, $AD = n + x$, and consequently, $PD = \sqrt{a^2 + x^2}$; whence, by similar Triangles, $DF : DP :: DA : DB = \frac{n+x}{x-c}\sqrt{a^2+x^2}$; which, by the Conditions of the Problem, is to be a *Minimum*, and therefore the Fluxion thereof $\frac{\overline{c+n} \times \dot{x} \times \overline{a^2 + x} + \overline{n + x} \times \overline{c - x} \times x\dot{x}}{\overline{c-x}|^2 \times \sqrt{a^2+x^2}}$ $= 0$; consequently $x^3 - 2cx^2 - ncx = a^2 \times \overline{n+c}$; from whence x may be determined, and consequently the Length of the Line (DB) passing through the given Point.

The same answered. R.

Let the Extremities of the two right Lines DB, db passing through the given Point P, and terminating in the two right Lines given by Position, be supposed indefinitely near to each other, and let DS and bm be so drawn, that each of the Angles dsD, bmB may be equal to the given Angle DAB; also, let Dr and bn be drawn perpendicular to dP and BP respectively, and let the Position of the other Lines, in the Figure, remain as before: Then if PF be put $= a$, $PC = b$, $FG = c$, $BC = x$, and $Bm = y$; by similar Triangles, it will be, $x : b :: y : \frac{by}{x} = bm$; $BP : DP :: x : a :: \frac{by}{x}$ $: \frac{aby}{x^2}$

$: \frac{aby}{x^2} = Ds$, and $x : b :: \frac{aby}{x^2} : \frac{ab^2y}{x^3} = ds$; alſo $a : c :: \frac{by}{x} : \frac{bcy}{ax} = nm$, and $a : c :: \frac{aby}{x^2} : \frac{bcy}{x^2} = rs$; therefore $Bm + mn = Bn = y + \frac{bcy}{ax}$, and $ds + sr = dr = \frac{ab^2y}{x^3} + \frac{bcy}{x^2}$: But when BD is the *Minimum*, it is manifeſt that Bn and dr muſt be equal; therefore $x^3 + \frac{bcx^2}{a} = cbx + ab^2$; whence it appears, if AS be perpendicular to AD, that $x^3 = ab^2$; becauſe $c = FG$ is then $= 0$.

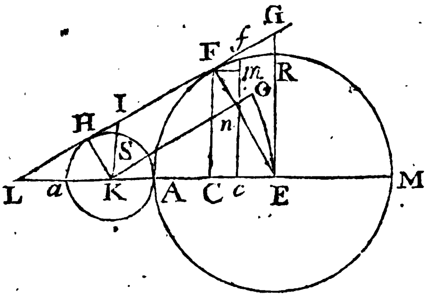
PROBLEM XVIII. *Anſwered.* R.

In order to which, it may not, perhaps, be improper to ſhew the Manner of inveſtigating the two following Series, as the former of them is applied to the Solution here propoſed. If the Radius of a Circle be r, and any Arch thereof z; the Sine correſponding to the ſame will be expreſſed by

$$z - \frac{z^3}{2 \cdot 3 r^2} + \frac{z^5}{2 \cdot 3 \cdot 4 \cdot 5 r^4} - \frac{z^7}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 r^6},$$

&c. and the Coſine by $r - \frac{z^2}{2r} + \frac{z^4}{2 \cdot 3 \cdot 4 r^3} - \frac{z^6}{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 r^5}$, *&c.*

For, let the Radius $FE = r$, Sine $FC = x$, Verſed-ſine $AC = y$, and the Arch $AF = z$; then, by ſimilar Triangles, $x : r :: \dot{y} : \dot{z}$, and $r - y : r :: \dot{x} : \dot{z}$; whence, from the former Proportion, $x\dot{z} = r\dot{y}$, and,



and, from the latter, $r\dot{z}-y\dot{z}=r\dot{x}$: But to obtain x in Terms of z and known Quantities, let x be assumed $=Az+Bz^3+Cz^5+Dz^7$, &c. and by Substitution $r\dot{y}=Az\dot{z}+Bz^3\dot{z}+Cz^5\dot{z}+Dz^7\dot{z}$, &c. also $r\dot{z}-y\dot{z}=rA\dot{z}+3rBz^2\dot{z}+5rCz^4\dot{z}+7rDz^6\dot{z}$, &c. whence, from the former of these Equations, $y=\frac{Az^2}{2r}+\frac{Bz^4}{4r}+\frac{Cz^6}{6r}+\frac{Dz^8}{8r}$, &c. and, from the latter, $r-y=rA+3rBz^2+5rCz^4+7rDz^6$, &c. consequently $r-y=r-\frac{Az^2}{2r}-\frac{Bz^4}{4r}-\frac{Cz^6}{6r}-\frac{Dz^8}{8r}$, &c. $=rA+3rBz^2+5rCz^4+7rDz^6$, &c. Now, since the Homology of the Terms depend not at all on the Coefficients, but altogether on the Powers of the variable Quantity z; the respective Terms of the two last Series may be compared with each other, and from thence will be had $A=1$, $B=\frac{-1}{2.3r^2}$, $C=\frac{1}{2.3.4.5r^4}$, $D=\frac{-1}{2.3.4.5.6.7r^6}$, &c. which being substituted in the Series assumed above, in Place of A, B, C, D, *&c.* there will result that expressing the Sine, and being substituted in either of the Series proved $=r-y$, there will result that expressing the Cosine. Q. E. D.

This

This being premised, if half the Length of the mixed Line be put $= S$, half the Sum of the two Diameters $= r$, the Arch $EO = z$, and the Sine corresponding the same $= x$; then will $Kn = HF = \sqrt{r^2 - x^2}$, $FE = \frac{r+x}{2}$, and $HK = \frac{r-x}{2}$; therefore $r : z :: \frac{r+x}{2} : \frac{rz+zx}{2r} = FR$, and $1 : \frac{p}{4} :: \frac{r+x}{2} : \frac{pr+px}{8} = MR$, p denoting the Periphery of a Circle whose Semi-diameter is Unity: In the same Manner it will appear that $HS = \frac{rz-zx}{2r}$, and $aS = \frac{pr-px}{8}$; consequently $aH + MF + FH = \frac{pr}{4} + \frac{zx}{r} + \sqrt{r^2 - z^2} = S$; but x, by what has been already proved, is $= z - \frac{z^3}{2.3r^2} + \frac{z^5}{2.3.4.5r^4} - \frac{z^7}{2.3.4.5.6.7r^6}$, &c. which being substituted above in Place of x, there will from thence arise an Equation, involving one unknown Quantity, whereby the Value of z may be determined, and from thence the Diameter of each Circle.

PROBLEM XIX. *Answered by the* Proposer.

The Equation defining the Curve, should have been $y = \frac{\overline{ax - x^2}|^{\frac{1}{2}} \times a^5 x^2}{\overline{a+x}|^9}$: Then the given Value of y, by converting the Denominator to a Series, &c. may be transformed to $\frac{\overline{a-x}|^{\frac{1}{2}} \times x^{\frac{1}{2}}}{a^4} \times \overline{x^3 - \frac{9x^4}{a} + \frac{9.10.x^5}{1.2.a^2} - \frac{9.10.11.x^6}{1.2.3.a^3}}$, &c. where if the Fluent of $\overline{ax - x^2}|^{\frac{1}{2}} \times \dot{x}$, when $x = a$, or the Area of the Semi-circle whose Diameter is a, be de-

K noted

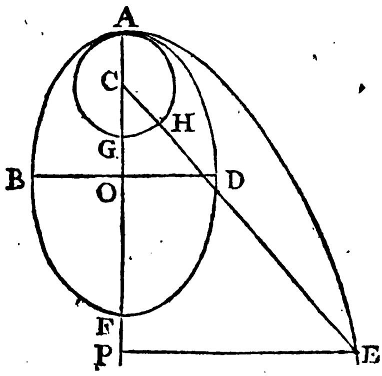
noted by Q, and the whole Series be supposed to be multiplied by $\dot{x}$, the Fluent of the first Term will be $\frac{3.5.7.3.Q}{6.8.10.12}$; of the second $-\frac{9.3.5.7.9.3.Q}{6.8.10.12.14}$, of the third $\frac{9.10.3.5.7.9.11.3.Q}{1.2.6.8.10.12.14.16}$, &c. and consequently the Fluent of the whole Expression will be $\frac{3.5.7.3.Q}{6.8.10.12}$ into $1-\frac{9.9}{14}+\frac{9.10.9.11}{1.2.14.16}-\frac{9.10.11.9.11.13}{1.2.3.14.16.18}$, &c. But the Sum of the Series $1-\frac{9.9}{14}+\frac{9.10.9.11}{1.2.14.16}$, &c. (by Prop. 1. *Summa. Series*, *Simpson*'s Dissertations) is found to be $\frac{419}{14336\sqrt{2}}$, which multiplied by $\frac{3.5.7.3.Q}{6.8.10.12}$ gives $\frac{7Q}{128}\times\frac{419}{14336\sqrt{2}}$ for the Area of the whole Curve, which therefore is to the Area of the Semicircle as 419 to $262144\sqrt{2}$.

PROBLEM XX. *Answered by Mr.* John Turner.

Let AHGA be the Earth, ADFBA the Trajectory, which the Body in Latitude 52° describes, AEC that described by the Body projected from under the Equator, and A the Place where the Bodies leave the Earth's Surface. It is found (in Page 148 of *Simpson*'s Fluxions) that the periodic Time of a Body describing a Circle, just above the Earth's Surface, by Means of its own Gravity, is 84′ 43″, therefore the Velocity with which the Body at the Equator is projected, is to the said circular Velocity, as $\frac{84' 43''}{38'}$, to 1, and therefore

the

the Velocity of the other Body, being to the Velocity of the first, as the Cosine of (52°) the given Latitude to Radius, it will be expressed by $\frac{84' 43''}{38'} \times \frac{\text{Cos. } 52°}{\text{Rad.}}$, the circular Velocity being denoted by Unity: Now let $\frac{84' 43'}{38'}$ be denoted by n, $\frac{84' 43''}{38'} \times \frac{\text{Cos. } 52°}{\text{Rad.}}$ by m, and the Earth's Radius AC by a, then m^2 being less than 2, the Body projected from the Parallel of 52° will describe an Ellipsis ABFDA, whereof the Semi-transverse AO $= \frac{a}{2-m^2}$, and the Semi-conjugate BO $= \frac{ma}{\sqrt{2-m^2}}$: Therefore, of different Bodies revolving round the same Center, the Squares of their periodic Times, being as the Cubes of the transverse Diameters, of the Section described, let the other Diameters be what they will, $AC^3 : AO^3 :: \overline{84' 43''}|^2 : \text{to } \frac{\overline{84' 43''}|^2}{\overline{2-m^2}|^3}$ the Square of the periodic Time in the Ellipsis ABFDA;



therefore $\frac{84' 43''}{\overline{2-m^2}|^{\frac{3}{2}}}$ is the periodic Time required, which divided by 38′ leaves $\frac{84' 43''}{38' \times \overline{2-m^2}|^{\frac{3}{2}}}$; therefore as 38′ is to the said Remainder, so is 360° to the Difference of Longitudes of the two Points of the same Parallel from whence the Body is projected and where it falls.

 Again,

Again, with regard to the other Body projected from under the Equator, the Curve AE which it describes, becaufe n^2 is greater than 2, will be an Hyperbola whofe Semi-tranfverfe is $\frac{a}{n^2-2}$, and the Semi-conjugate $\frac{na}{\sqrt{n^2-2}}$; (fee *Simpfon*'s Fluxions, Page 154) and the Hyperbolical Sector ACEA defcribed in 6 Hours, will be to the Area of the Circle AHGA defcribed in 38′, as 6 Hours to 38′, and therefore will be expreffed by $\frac{\text{AHGA} \times 360'}{38'}$ $= \frac{a^2 \times 360' \times 3.141592, \&c.}{38'} = \frac{ra^2}{2}$; but the Area ACEA, fuppofing the Ordinate PE to be expreffed by az, will be alfo expreffed by $\frac{\overline{n^2-1} \times a^2 z}{2 \times \overline{n^2-2}}$ $-\overline{\frac{na^2}{2 \times n^2-2}}\big|^{\frac{3}{2}} \times$ Hip. Log. $\sqrt{a^2 z^2 + \frac{n^2 a^2}{n^2-2}} + az$ $\times \frac{\sqrt{n^2-2}}{na} = \frac{ra^2}{2}$, or $\frac{\overline{n^2-1} \times z}{n^2-2} - \overline{\frac{n}{n^2-2}}\big|^{\frac{3}{2}} \times$ Hip. Log. $\sqrt{z^2 + \frac{n^2}{n^2-2}} + z \times \frac{\sqrt{n^2-2}}{n} = r$; therefore, putting $n^2 - 1 = b$, $\frac{n}{\sqrt{n^2-2}} = c$, and $\frac{1}{n^2-2} = d$, it will become $bdz - cd \times$ Hip. Log. $\frac{\sqrt{z^2+c^2}+z}{c} = r$, or $z = \frac{r}{bd} + \frac{c}{b} \times$ Hip. Log. $\frac{\sqrt{z^2+c^2}+z}{c}$; whence the Diftance of the Body from the Earth's Center may be determined.

This

This Problem was, alſo, anſwered by *Tycho Oxonienſis*, who makes the periodic Time of the Body projected from the Parallel of 52°, to be 1 Day 9 Hours 44′ 52″, and that projected from under the Equator, to be 186947 Miles diſtant from the Earth's Center at the End of the propoſed Time.

A COLLECTION OF PROBLEMS TO BE Anſwered in the next NUMBER.

PROBLEM XXI. *L.*

IF to enjoy the Benefit of an Eſtate for 23 Years after the Expiration of eight Years, be worth 400*l.* preſent Money; what will the ſaid Eſtate be worth for 21 Years after the Expiration of 10 Years, at the Rate of 5*l.* for every 100 Yearly.

PROBLEM XXII. *L.*

A Traveller benighted, ſees before him two Lights, and looking back diſcovers three others, in a right Line with the former; and judges the Quantity of Light received from the former Place, to

to be to that received from the latter, in the Ratio of 3 to 4; then proceeding forwards 400 Yards, he finds the Ratio of Light there to become as 5 to 3: The Question is, suppofing all the Lights to be equal among themfelves, what Diftance he was from each Set of Lights, at the two Places of Obfervation.

PROBLEM XXIII. *A* Theorem *to be demonftrated.*

As the leffer of the two Sides, including any propofed Angle, of a Triangle, is to the greater, fo is Radius, to the Tangent of an Arch or Angle: And as Radius, is to the Tangent of the Excefs of the faid Angle above half a right Angle, fo is the Tangent of half the Sum of the oppofite Angles, to the Tangent of half the Difference of the fame Angles.

PROBLEM XXIV. *Thomas Hulme*, London.

The Line bifecting the Bafe, the Difference of the Sides, and the Difference of the Angles at the Bafe, of any plane Triangle being given; to determine the Triangle.

PROBLEM XXV. *John Turner*, London.

The Bafe of any plane Triangle, the vertical Angle and the Side of the infcribed Square being given; to conftruct the Triangle.

PROBLEM XXVI. *R.*

To defcribe a Circle, through a given Point, which fhall touch a right Line given in Pofition, and alfo another Circle given in Magnitude and Pofition.

PROBLEM XXVII. *R.*

To describe a Circle, through two given Points, that shall touch another Circle given in Magnitude and Position.

PROBLEM XXVIII. *N.*

One Side and all the Angles, made by the adjacent Sides and the two Diagonals, of any Trapezium being given, except the two Angles contained between them and the given Side; to describe the Trapezium: And that without assuming any similar Figure.

PROBLEM XXIX. *John Moor.*

One Side and the Distances from the Center of Gravity to each Angle, of any Trapezium being given; to determine the Trapezium.

PROBLEM XXX. *Walter Trott*, London.

A Ship sailed from the *Lizard* (Lat. 50° 00′ N.) S. W. 20 Miles, and was there taken by a *French* Privateer; who took away their Compass, and afterwards kept them Company steering between the South and East until, as the Privateer told them, the *Lizard* bore due North, and then the Privateer left them; and they continued the same Course, as near as possible, and run by the Log. 25 Miles; then they spoke with a Man of War, who informed them the *Lizard* bore N. W. by W. required the Course and Distance sailed in Company with the Privateer; also their Distance from the *Lizard*, with the Latitude the Ship is in, and her Departure from the Meridian.

PRO-

PROBLEM XXXI. *John Turner*, London.

Of all the spheroidical Casks having the same given Diagonal, to find that whose Content is the greatest.

PROBLEM XXXII. *R.*

Of all the conic Parabolas passing through four given Points, to determine the least.

PROBLEM XXXIII. *Hamilcar.*

To determine such a Part of a spherical Superficies, which can be illuminated in its farther Part, by Light coming from a great Distance, and which is refracted by the nearer Hemisphere.

PROBLEM XXXIV. *R.*

If a Curve be supposed to revolve upon its principal Axe, and thereby generate a Solid, the Frustum of which is of such a Nature, that the Difference between the two Diameters multiplied by any fractional Number less than Unity, and the Result taken from the greater, produces the Diameter of a Cylinder of equal Magnitude having the same Length as the Frustum; then it is required to find the Equation of the Curve from whence it is generated.

PROBLEM XXXV. *T. G——t.*

If the Relation between the Abscis, and the Ordinate of a Curve, be expressed by $yx = \overline{a+cx^n}|^m \times dx^{pn}$; the Area, supposing x the Abscis, y the Ordinate, and $a + cx^n = v$, will

be

be expressed by $\frac{dv^m x^{pn}}{pn} \times \overline{1 - \frac{m}{p+1} \times \frac{cx^n}{v} + \frac{m.m-1}{p+1.p+2} \times \frac{c^2 x^{3n}}{v^2} - \frac{m.m-1.m-2}{p+1.p+2.p+3} \times \frac{c^3 x^{3n}}{v^3}}$, &c. Required the Investigation.

Problem XXXVI. *R.*

Let it be required to find the Sum of the Series $\frac{m-1.n-1.p.-1.}{r} + \frac{m-2.n-2.p-2}{r^2} + \frac{m-3.n-3.p-3}{r^3}$, &c. continued to m Terms, and to shew the Investigation of the same.

Problem XXXVII. *Thomas Leigh*, London.

If a Cistern, in Form of the Frustum of a Sphere, whose Axis, or Depth is 10 Feet, and the Diameter at the Top 30 Feet, be supplied with Water running uniformly from a Cock that can fill it in two Hours, and at the Bottom be another Cock, out of which the whole Cistern may be evacuated in one Hour, the former Cock being stopped; it is required to find, if the Cistern was full, and both the said Cocks open, in what Time the Surface would descend four Feet, and the greatest Distance it could possibly descend.

Problem XXXVIII. *R.*

To find, supposing the Earth's Rotation about its Axis was entirely to cease, how much Pendulums would gain in 24 Hours in the Latitude 51° 30′.

PROBLEM XXXIX. *John Turner*, London.

Suppose the Earth, instead of being nearly spherical, was to revolve about its Axis, in such a Time, that the Equatorial Diameter should be just double the Polar Diameter; it is proposed to find in what Time, a Body in the Latitude 50 Degrees, falling every where in the Direction in which it gravitates without Interruption or Resistance, would descend to the Earth's Center.

PROBLEM XL. *T.*

To investigate the Path, which a fixed Star by Means of the Aberration would appear to describe, if the Earth instead of revolving in an Ellipsis, was to move in a Parabolic or Hyperbolic Orbit.

The End of NUMBER II.

THE

Mathematician.

DISSERTATION III.

Upon the Progreſs and Improvement of GEOMETRY.

IN Purſuance of the metaphyſical Definitions of Time, Space, and Velocity in our 2d Diſſertation mentioned, Dr. *Barrow* proceeds to ſhew, how the various Magnitudes, about which Geometry is converſant, may be conceived to be generated, conſiſtently with their general Properties, either by *ſimple Motions*, or by *compound Motions*, or by the *Concurrence of Motions:* Firſt, he enquires what are the Hypotheſes and Effects ariſing from ſimple Motions; of theſe there are two Kinds, *Progreſſive*, and *Circumrotatory*. The

chief

chief Hypotheses fram'd by Mathematicians about these kind of Motions, being of the greatest and most frequent Use to the Formation of Magnitudes, are these; *viz.* that a Point may move directly on, as long as you please from an assign'd Term, in a right Line, by which Motion it is evident, a right Line, being the Track of a Point, is describ'd (which Geometrical Point having no Dimensions, nor being material, but purely ideal, neither will the Line which is the trace of it, have any;) that a right Line may so move along any other Line straight or crooked, as in the mean while, perpetually to keep a parallel Situation; *i. e.* in any Moment of Time to have a Situation parallel to what it had in any preceding Moment. Also, that any Line, definitely or indefinitely, extended, may proceed forwards with a direct Motion, likewise parallel to itself; that is, that every Point thereof may describe right Lines. Thus by two Motions equable and uniform, *Parallelograms*, and *prismatic* and *cylindric* Superficies and Solids are suppos'd to be form'd: One of the Lines by whose Motion a Magnitude is described, is called the *Generative Line*; the other according to which the former moves, or on which it stands, the *Directive*; because the Course of the Line moved is governed by it, or accommodated to it.

The other kind of simple Motion, made use of in Mathematics, is Circumrotation, and is made when something of the Magnitude moved, (as suppose any Point of a Line, or Line of a Superficies) remains fixed and immoveable, while the whole Magnitude remaining as it were tied and bound to the same, is carried about according to any assign'd Direction. The most general Property of which Motion is, that all Points of the moveable Magnitude, while they move transversly in any one Plane, do every one describe the Peripheries of Circles; and indeed all moving in one and the same Plane

passing

passing thro' a fixed Point, are parallel, and concentric, and similar to each other: But those in different Planes are similar, or not so, according as the arbitrary Diversity of Hypotheses requires: And doubtless Circumrotation is what Nature itself conceives and follows, whereby Magnitudes are kept to their several immoveable Holds, and hindered from flying off in right Lines which they naturally would do, as appears in the Motion of Pendulums, and Bodies in Slings; or even when any Object being resisted cannot easily keep in a strait Path; as appears to be the Case of the Motions of *Wheels*, *Whirlpools*, *Whirlwinds*, *and perhaps of the Stars themselves*. The principal Hypotheses of these Motions are, 1st. That a right Line in a Plane can be moved about any Point fixed in it; by which Motion it is evident all Points of the moving Line describe the Peripheries of Circles, all parallel and similar to one another. From which Generation and the Doctrine of Indivisibles we infer that the Areas of Circles, and circular Sectors are made up of similar and concentric circular Peripheries, as many in Number, as there are Points in the Radius; by means whereof may be easily deduced the well-known Mensuration of the Area of a Circle.

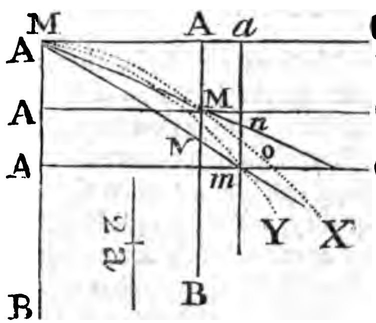
2dly. That any right Line indefinitely extended, having one Point thereof fixed, may revolve about any given Line, either a Curve, or consisting of right Lines constituted in some other Plane, so as always to touch this Line, or as it were stick to it; by this Motion will a *pyramidal or conical Superficies* be produced. But the most usual Manner among Mathematicians of producing Bodies, is that which is peculiarly called by the Name of Rotation, and is performed by supposing any Line or plane Superficies to revolve about an immoveable Line as an Axis. Thus a *Spherical Superficies* is produced from the Motion of *half the Circumference of a Circle* a-

bout the *Diameter*; a Sphere by the *Rotation* of the *Semicircle* itself; a *cylindrical Superficies* by the Motion of a right Line about a Line parallel to itself; a *right Cylinder*, by the Motion of a right-angled Parallelogram about one Side; a *conical Superficies* by the Motion of one Side of a right-lin'd Angle about the other; a Cone itself by the Rotation of a right-angled Triangle about one of its Legs; and after this Manner may be generated *innumerable solid Magnitudes with their curve Superficies, either the Wholes or their Parts, Frustums, Tubes, Rings.* The chief Property of which Motion is, that every particular Point of the Magnitude moved about, does describe circular Peripheries (completely effected, when the Revolution is quite performed, or the moveable Magnitude returned to its first Situation, all described at the same time being similar) whose Centers are all in the said Axis, and Radii right Lines perpendicular to it: Or, that all right Lines situate in the moveable Magnitude being perpendicular to the Axis do describe Circles, when the Revolution is finished, or similar circular Sectors, in the same Time.

Thus far concerning Simple Motions; in the next Place the Dr. treats of compound and conspiring Motions, in describing the Effects whereof, the *Velocities* wherewith the simple Motions are performed, must for the most part be considered; tho' in *Generations* by simple Motions, they are not in the least regarded: For the same Magnitude may be produced by the same simple Motion, whether swifter or slower it matters not, altho' not in the same Time. But in a Generation by compound Motion the same Ways of Lation or Direction remaining, as the Velocity of one or several varies, there arises Magnitudes not only altering in Species, but Quantity too, or at least perpetually differing in Position.

There

There is no kind of Magnitude, *viz.* no Line, Superficies or Solid, but what may be conceiv'd to be generated by ſtrait Motions; and all Lines lying in the ſame Plane, may be generated by the parallel Motion of a right Line and a Point moving at the ſame time along the ſame; but theſe Motions ought to be ſo temper'd, as the particular Nature of the Curve requires; nor may any regard be had how variable a Velocity you attribute to one of the Motions, provided that of the other be duly accommodated to it. For Example, if the right Line BA be always carried along the right Line AC parallel to itſelf with any equable or unequable Motion (increaſing, decreaſing, or varying its Velocity in any Manner imaginable,) and at the ſame time any Point M be moved along it, ſo that the Motion of the Point or Space paſſed over in every Inſtant of Time, be proportionable to the Motion of the right Line; there will be a right Line produced: *i. e.* if it be granted that always during this Deſcription AA : A*a* :: AM : *an*, then the 3 Points A, M, *n*, will lie in a right Line *per* 32 E. 6.



If the Motion of the right Line AB remains the ſame as to Velocity, but the Velocity of the Point M be increaſed, thus indeed the Point M will arrive at *m*, but another right Line AN*m* different in Poſition from the other will be produced.

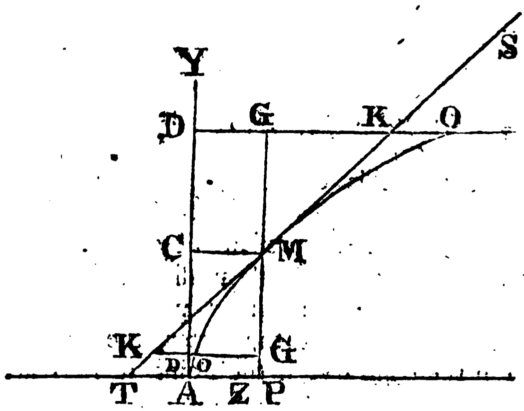
Alſo

Also if these Motions are so related, that (taking a determinate right Line $2a$ and calling $am=x$ and Aa or $Am=y$) the Rectangle under the Difference of $2a$ and am (moved thro' by the moveable Point in the right Line AB) into am, shall always have to the Square of Aa (moved thro' by the progressive Line AB in the same time) a given Ratio (suppose that of $2a$ to $2p$) then will an Ellipsis or Circle be described; a Circle when the proposed Ratio, is a Ratio of Equality, and the Angle BAC a right Angle; and an Ellipsis, when it is otherwise. You may express the very same Truth analytically by an Equation shewing the Relation these Lines (which are the Effects of the Motions above mentioned) always bear to one another; for if this Proportion always holds true, *viz.* $\overline{2a-x}\times x : yy :: 2a : 2p$, then whenever $a=p$, at the same time will $2ax-xx=yy$ which is the known Expression for a Circle: The above Proportion expresses a well-known and fundamental Property of the Ellipses, when $2a$ represents the transverse Diameter, and $2p$ the Parameter, and may be resolv'd into this Equation $\frac{a}{p}yy=2ax-xx$. But if these Motions are such, that the Rectangle under the Sum of the Lines $2a$ and ao into ao, always is in the same Proportion to the Square of Aa, then will an *Hyperbola* be produced by this compound Motion: Being *equilateral*, when the said Ratio is a Ratio of Equality, and the Angle BAC right; but if not will be of another kind, according to the Quantity of the assign'd Ratio, whose transverse Diameter will be $=2a$, situate in BA extended from the Vertex A, the contrary way to B. Also if the Rectangle under $2a$ and am, which is mov'd over by the Point M has perpetually the same Ratio to the Square of Am, there will be a parabolic Line describ'd; for the Equation will then be $2ax=yy$. In the

first of these Cases, *viz.* that concerning the right Line, if the transverse Motion along AC be suppos'd uniform, the descending Motion thro' AB shall be uniform too: In the 2d and 3d if the Motion along AC be uniform, the descending Motion will perpetually increase; and supposing the same thing with regard to the last Case, wherein the Parabola is described; the Point M will perpetually encrease equably in Velocity. Much after the same Manner, may any other Line be conceiv'd to be generated by some such like Composition of Motion.

In the next Place, the Dr. subjoins some of the general Properties, which flow from Lines generated by an uniform, progressive Motion, and a descensive Motion continually increasing; where by reason of the Uniformity of the Motion along AC, and its Parallels, it may represent the Time of the Motion, and the Parts of the one, the Parts of the other.

The 11th Property, being a very important one in its Consequences, is here exhibited with its Demonstration; it is this;



Let the Tangent TM be a Tangent in M to the Curve AOM, then the Velocity of the descending Point

Point in M, is to the equable Velocity with which the right Line AZ moves, as the right Line TP is to PM. Or the Velocity of the descending Point which it has at M, is equal to the Velocity by which the Subtangent TP would be described with an uniform Motion, in the same Time as the right Line AZ is moved uniformly thro' AC or PM. For take the Point K any where in the Tangent, and thro' the same draw the right Line KG, meeting the Curve in O, and the Parallels AY, PG, in the Points D, and G. Then because the Tangent TM, is conceived to be described by two uniform Motions, the one of the right Line TZ carried thro' the Parallels AC or PM, and the other of the descending Point from T, thro' TZ; and since one of these Motions thro' AC or PM is the same with that whereby the Curve is described, it is therefore common both to the Tangent and to the Curve; when TZ is in the Situation KG, AZ will be in the same: Therefore when the Point descending from T is in K, the Point descending from A, will be in O, the Intersection of the Curve, and right Line KG. Now if the Point K be suppos'd below the Point of Contact towards T, because then OG is less than KG, it is evident, that the Velocity of the descending Point, whereby the Curve is described, in the Point O of the Curve, is less than the Velocity of the uniform descending Motion, wherewith the Tangent is effected; because the former always increasing makes a less Space in the same Time (which Time is represented by GM, than the latter which does not encrease at all, but perpetually continues the same; the former passing over only the right Line OG, but the latter, the right Line KG. On the contrary, if the Point K falls above the Point of Contact towards S; because OG is then greater than KG, it is evident that the Velocity of the descending Point (whereby the Curve is described)

in

in the Point O, is greater than the Velocity of the uniform descending Motion, whereby the Tangent is effected; because the former Motion continually increasing, in the same Time thro' OM, makes a greater Space OG, than the latter not at all increasing, but continuing perpetually, does, *viz.* the Space KG. Therefore because the Velocity of the Point describing the Curve, in any Point of the Curve below the Point of Contact towards A, is less than the Velocity of the Motion thro' TP; but in any Point of the Curve above the Point of Contact, is greater than the same; it is manifest therefore that in that very Point it is equal to it. W. W. D.

Hence it follows that the Velocities of the descending Point in any two assigned Points of the Curve, are to one another reciprocally in a Proportion compounded of the Ratio's of the Ordinates and of the Subtangents.

Likewise hence may be inferr'd a general Solution of that Problem which *Galileo* esteemed so much, and employed so much Time about, *viz.* Any Parabola being given whose Vertex is A, it is required to find some Point aloft, from whence if a heavy Body falls to A, and the Impetus there received be at that Point turn'd into a horizontal one, the proposed Parabola may be described. This is no other than to determine the particular Velocities of the uniform horizontal or transverse Motion, and an equal increasing descensive one, by the Composition whereof the given Parabola is described. Thus by metaphysical and plain Geometry without the Assistance of Analytics, are we arrived at the very door of the higher Geometry; a compleat Entrance into which must be administred unto us, by that effectual Key, the Method of Fluxions; to which Method the Property last above demonstrated bears a great Affinity, for, to find the Ratio of the Velocities of the Motions by which a Magnitude is conceived to

be generated; is the very same thing as, to find the Fluxions thereof: And the particular Case abovementioned, *viz.* the finding the Fluxions of the Abscissa and Ordinate of a Geometrical Curve, is nothing but finding two finite Magnitudes which have the same Ratio, *i. e.* the Subtangent to the Ordinate; and is generally the first Problem in a Teatise upon Fluxions.

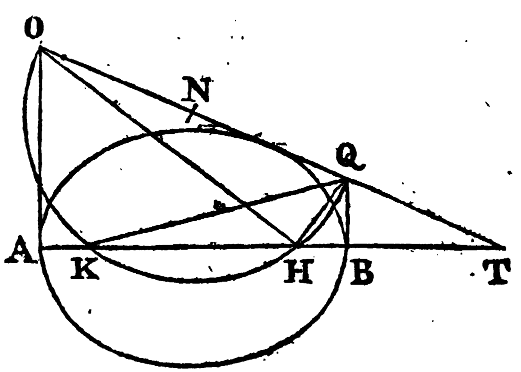
The Remainder of this in our next.

CONIC SECTIONS.

The Properties of the ELLIPSE *continued.*

PROPOSITION XXXI.

IF Perpendiculars drawn from the Vertices cut any Tangent; then the Part of the Tangent intercepted between the Intersections, will be the Diameter of a Circle, the Periphery of which shall pass through the Foci.



DEMONSTRATION.

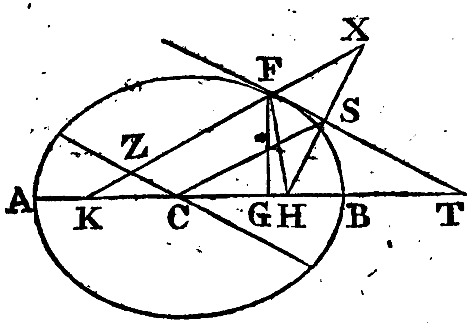
By Prop. 24, $AQ \times BQ = AK \times KB$; therefore $AO : AK :: KB : BQ$; but the Angles OAK and QBK are right Angles; therefore (by Eu. 6. 6.) the Triangles OAK and BQK are similar, and the Angle $AOK = QKB$; but the Angle $AKO + AOK = AKO + QKB =$ to a right Angle, and consequently (by Eu. 13. 1.) the Angle $OKQ =$ to a right Angle; therefore (by Eu. 31. 3.) OQ is the Diameter of a Circle, the Periphery of which will pass through K one Focus: In like manner the Angle OHQ may be proved a right Angle, and therefore the Periphery of the foresaid Circle, will pass through H the other Focus. *Q. E. D.*

COROLLARY.

If OQ be bisected in N; then $NO = NQ = NK = NH$.

PROPOSITION XXXII.

If, from either Focus a right Line be drawn thro' the Point of Contact, and continued till it become equal to the transverse Axe, and from the Extremity thereof a right Line be drawn to the other Focus; then the Distance between the Center, and the Intersection of the last Line with the Tangent, will be equal to half the transverse Axe; that is $CS = CB$.



DE-

DEMONSTRATION.

In the Triangles HCS, HKX, the Angle KHX being common, KC=CH, and HS=SX (by Prop. 26.) therefore (by Eu. 6. 6.) the Triangles are similar, and consequently CS parallel to KX; also CS= $\frac{1}{2}$XK= $\frac{1}{2}$AB=CB. *Q. E. D.*

PROPOSITION XXXIII.

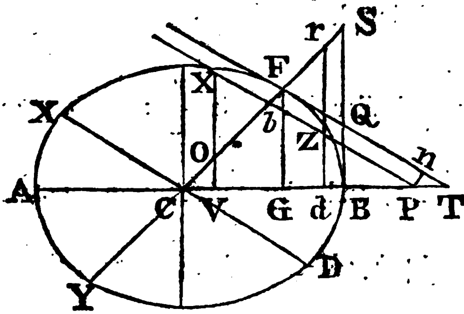
If, from the Focus, a right Line be drawn to the Point of Contact, and another through the Center parallel to the Tangent, then the Difference between the Point of Contact and the Intersection of these Lines will be equal to half the transverse Axe.

DEMONSTRATION.

Draw CS parallel to KF; then will the Figure ZCSF be a Parallelogram, and ZF=CF=(by Prop. 32.) BC. *Q. E. D.*

PROPOSITION XXXIV.

If to the Tangent drawn to the Vertex of any Diameter, a right Line be drawn parallel, the Part of that Line which lies within the Curve will be bisected by the Diameter; that is, *b*X=*b*Z: Also the Triangle BCS= the Triangle *d*PZ=*dr*C= the Triangle XPV+OVC.



DE-

DEMONSTRATION.

Let $dZ=y$, $dP=c$, $dC=n$, $BS=r$, $dr=p$, $XV=Y$; $CV=g$, $VO=q$, $PV=b$, $dB=x$, and $BV=X$; then by similar Triangles $\frac{y}{c}=\frac{FG}{GT}=$ (by Prop. 22.) $\frac{CG}{GF}\times\frac{p}{t}$; but $\frac{CG}{GF}=\frac{t}{2r}$, and therefore $\frac{y}{c}=\frac{2t}{2r}\times\frac{p}{t}$; which being first divided by $\frac{p}{t}$, and then multiplied by cry, gives $\frac{tcy}{2}=ry^2\times\frac{t}{p}$; but $\frac{ty^2}{p}=tx-x^2$, (by Prop. 2.) therefore $\frac{tcy}{2}=r\times\overline{tx-x^2}$, and $\frac{tcy}{2}+rn^2$ $=r\times\overline{tx-x^2}+rn^2=r\times\overline{tx-x^2+n^2}$; but (by Eu. 5. 2.) $tx-x^2+n^2=\frac{t^2}{4}$; therefore $\frac{tcy}{2}+rn^2=\frac{rt^2}{4}$, and each Part being divided by $\frac{t}{2}$ gives $cy+\frac{2rn^2}{t}=\frac{rt}{2}$; but from similar Triangles $\frac{t}{2}:r::n:p$; therefore $\frac{2rn}{t}=p$, and, by Substitution, $cy+pn=\frac{rt}{2}$; or $dP\times dZ+dr\times dC=BS\times BC$; that is, the Triangle

BCS=

BCS= the Triangle $dPZ+drC$. Again, by similar Triangles, $\frac{Y}{b}=\frac{FG}{GT}=$(by Prop. 22.) $\frac{CG}{GF}\times\frac{p}{t}$; but $\frac{CG}{GF}=\frac{g}{q}$, and therefore $\frac{Y}{b}=\frac{gp}{qt}$; which being first divided by $\frac{p}{t}$, and then multiplied by bqY, gives $gbY=\frac{tqY^2}{p}$; but $tX-X^2=\frac{t}{p}\times Y^2$; therefore $gbY=q\times\frac{p}{tX-X^2}$, and $gbY+qg^2=q\times\overline{tX-X^2+g^2}$; but (by Eu. 5. 2.) $tX-X^2+g^2=\frac{t^2}{4}$; therefore $gbY+qg^2=\frac{qt^2}{4}$, and each Part divided by g, gives $bY+qg=\frac{qt^2}{4g}$; but from similar Triangles $g:q::\frac{t}{2}:r$; therefore $\frac{tq}{2g}=r$, and, by Substitution $bY+qg=\frac{rt}{2}$; that is, $PV\times VX+VO\times CV=CB\times BS$; or the Triangle $XPV+OCV=$ the Triangle BCS=, by the first Part, the Triangle $dPZ+drC$; whence if from each Part of the last Equation, be taken the Triangle PbC, there will remain the Triangle bZr equal and similar to the Triangle ObX, and therefore $Xb=bZ$. *Q. E. D.*

PRO-

Proposition XXXV.

The ſame Things being ſuppoſed as in the laſt Propoſition; the Triangle BSC= the Triangle CFT; alſo the Trapezium *dr*BS= the Triangle P*d*Z; the Trapezium *b*PTF=the Triangle *b*Z*r*, and $\overline{FT+bP}\times bF=Zb\times br$.

Demonstration.

From ſimilar Triangles BS : FG :: BC : GC :: (by Prop. 14.) CT : BC; therefore BS×BC=FG×CT; or the Triangle BSC= the Triangle CFT= (by Prop. 34.) the Triangle P*d*Z+*dr*C; from the firſt, and third Equation take the Triangle *dr*C, and there remains the Trapezium *dr*BS= the Triangle P*d*Z: Alſo from the ſecond and third Equation take the Triangle P*b*C, and there remains the Trapezium *b*PTF=the Triangle *b*Z*r*; therefore (by the Lemma to Prop. 11. of the Parabola) $\overline{FT+bP}\times bF=Zb\times br$. *Q. E. D.*

Lemma.

The ſame Things being ſtill ſuppoſed $bY\times nT=Zb\times br$.

Demonstration.

For YC=FC : *bc* :: FT : *b*P; whence, by Compoſition, YC+*bc* : *bc* :: FT+*b*P : *b*P, and, by Alternation, YC+*b*C (=*b*V) : FT+*b*P :: *b*C : *b*P :: *np* (*b*F) : *n*T; therefore $bY\times nT=\overline{FT+bP}\times bF=$ (by Lemma to Prop. 11. Part 1.) $Zb\times br$.

Definition.

Definition. If $FS : FQ :: br : bZ :: 2FT : P$, the Parameter belonging to the Diameter FY; then $P = bZ \times \frac{2FT}{br}$, and,

PROPOSITION XXXVI.

As any Diameter, is to its Parameter, ſo obtained, ſo is the Rectangle of any Abſciſſas of that Diameter, to the Square of the Ordinate which divides them; that is, if D be put for the Diameter FY, x for the Abſciſſa bF, and y for the Ordinate $bX = bZ$; then $D : P :: Dx - x^2 : y^2$.

DEMONSTRATION.

By the Definition $P = y \times \frac{2FT}{br}$; therefore $\frac{P \times \overline{Dx - x^2}}{D} = \frac{y \times 2FT \times \overline{Dx - x^2}}{br \times D} = \frac{y}{br} \times \overline{Dx - x^2} \times \frac{2FT}{D}$; but by ſimilar Triangles $\frac{nT}{nP} = \frac{2FT}{D} = \frac{nT}{x}$, and therefore by Subſtitution $\frac{P \times \overline{Dx - x^2}}{D} = \frac{y}{br} \times \overline{D - x} \times nT =$ by the preceding Lemma $\frac{y}{br} \times y \times br = y^2$; whence $D : P :: Dx - x^2 : y^2$.

PROPOSITION XXXVII.

As any Parameter, is to its correſpondent Diameter, ſo is the Square of its Conjugate, to the Square of the Diameter; that is, $P : YF :: \overline{FY}^2 : \overline{XD}^2$.

DEMONSTRATION.

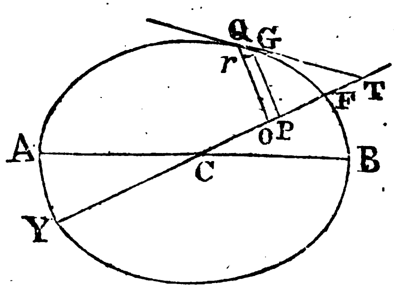
In this Case $x=\frac{D}{2}$, and $y=\frac{C}{2}$; therefore by the last Prop. $D : P :: \frac{D^2}{4} : \frac{C^2}{4} :: D^2 : C^2$; or $P : YF ::$ $\overline{FY}^2 : \overline{XD}^2$.

COROLLARY.

Hence any conjugate Diameter, is a mean Proportional, between the Diameter to which it is conjugate, and the Parameter of that Diameter: For by this Prop. $DP=C^2$; therefore $D : C :: C : P$.

PROPOSITION XXXVIII.

If a Tangent cut any Diameter, and from the Point of Contact an Ordinate be drawn to that Diameter; then it will be: As the Distance between that Ordinate and the Center, is to the Abscissa; so is the Diameter less by the Abscissa, to the Subtangent on the Diameter continued; that is, $CP : PF :: YP : TP$.



Let GQ an indefinite small Part of the Curve, be continued till it cut the Diameter produced in T; draw

draw the Ordinate GP, and parallel to it QO and Gr parallel to the Diameter FY: Put YF$=$D, FP$=x$, GP$=y$, G$r=n$, Q$r=m$, and FT$=a$; then YP$=$D$-x$, OY$=d-x-n$, OF$=x+n$, QO$=y+m$, and PT$=a+x$; then, by similar Triangles, $m:n::y:a+x$; therefore $a+x=\frac{ny}{m}$, and (by Prop. 36.) $D:P::Dx-x^2:y^2$; also $D:P::Dx-x^2+Dn-2nx-n^2:y^2+2my+m^2$; therefore $Dy^2=PDx-Px^2$, and $Dy^2+2mDy=PDx-Px^2+PDn-2nPx$; wherefore $PDx-Px^2+PDn-2nPx-2mDy=Dy^2=PDx-Px^2$, and $PDn-2nPx=2mDy$; consequently $n=\frac{2mDy}{PD-2Px}$; but $a+x$ being $=\frac{ny}{m}$; therefore $a+x=\frac{2mDy}{PD-2Px}\times\frac{y}{m}=\frac{Dy^2}{P}\times\frac{2}{D-2x}=$ (because by Prop. 36. $\frac{Dy^2}{P}=Dx-x^2$) $\overline{Dx-x^2}\times\frac{2}{D-2x}=\frac{2Dx-2x^2}{D-2x}=\frac{Dx-x^2}{\frac{1}{2}D-x}$; that is $\frac{D}{2}-x:x::\frac{1}{2}D-x:x+a$; or CP : PF :: YP : PT. *Q. E. D.*

PROPOSITION XXXIX.

If a Tangent intersect any Diameter, and from the Point of Contact, an Ordinate be drawn to that Diameter; then it will be: As the Semi-diameter less by the Abscissa, is to the Semi-diameter, so is the Semi-diameter, to the Semi-diameter added to the external Part of the Diameter, produced to the

Interſection of the Tangent: that is, CP : CF : : CF : CT.

DEMONSTRATION.

Since CP+PT=CT; CP$=\frac{D}{2}-x$, and PT (by the laſt) $=\frac{Dx-x^2}{\frac{1}{2}D-x}$; alſo CT$=\frac{D}{2}+a=\frac{D}{2}-x+\frac{Dx-x^2}{\frac{1}{2}D-x}=\frac{\frac{1}{4}D^2}{\frac{1}{2}D-x}$; therefore $\frac{D}{2}-x:\frac{D}{2}::\frac{D}{2}:\frac{D}{2}+a$; or CP : CF : : CF : CT.

PROPOSITION XL.

The ſame Things being ſuppoſed as in the laſt, it will be: As the Semi-diameter leſs by the Abſciſſa, is to the Semi-diameter; ſo is the Abſciſſa, to the external Part of the Diameter produced to the Interſection of the Tangent; that is, CP : CF : : PF : FT.

DEMONSTRATION.

By Prop. 39. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D-x}=\frac{1}{2}D+a$; therefore $\frac{D^2}{4}=\frac{D^2}{4}+\frac{Da}{2}-\frac{Dx}{2}-ax$, and $a=\frac{\frac{1}{2}Dx}{\frac{1}{2}D-x}$; that is, $\frac{D}{2}-x:\frac{D}{2}::x:a$; or CP : CF : : PF : FT. *Q. E. D.*

PROPOSITION XLI.

As the Semi-diameter leſs by the Abſciſſa, is to the Semi-diameter; ſo is the Diameter leſs by the Abſciſſa, to the Diameter added to the external Part of the Diameter produced to the Tangent; that is, CP : CF : : YP : YT.

DE-

DEMONSTRATION.

By the 40th, $a=\frac{\frac{1}{2}Dx}{\frac{1}{2}D-x}$; therefore $D+a=D+\frac{\frac{1}{2}Dx}{\frac{1}{2}D-x}=\frac{\frac{1}{2}D^2-\frac{1}{2}Dx}{\frac{1}{2}D-x}$; that is, $\frac{1}{2}D-x:\frac{1}{2}D::D-x:D+a$, or, CP : CF :: YP : YT. *Q. E. D.*

PROPOSITION XLII.

As the Diameter leſs by the Abſciſſa, is to the Diameter added to the external Part; ſo is the Abſciſſa, to the external Part of the Diameter produc'd to the Tangent; that is, YP : YT :: PF : FT.

DEMONSTRATION.

By the 40th, $\frac{1}{2}D-x:\frac{1}{2}D::x:a$; and (by the 41ſt) $\frac{1}{2}D-x:\frac{1}{2}D::D-x:D+a$; therefore by Equality $D-x:D+a::x:a$. Or, YP : YT :: PF : FT.

PROPOSITION XLIII.

As the Semi-diameter added to the external Part, is to the Semi-diameter; ſo is the external Part, to the Abſciſſa; that is, CT : CF :: FT : FP.

DEMONSTRATION.

By the 40th, $\frac{1}{2}Da-ax=\frac{1}{2}Dx$; therefore $x=\frac{\frac{1}{2}Da}{\frac{1}{2}D+a}$; that is, $\frac{1}{2}D+a:\frac{1}{2}D::a:x$. Or, CT : CF :: TF : PF.

PROPOSITION XLIV.

As the Semi-diameter added to the external Part, is to the Diameter added to the external Part; ſo is the

the external Part, to the Subtangent; that is, CT : YT :: FT : PT.

Demonstration.

By the 43d, $x=\frac{\frac{1}{2}Da}{\frac{1}{2}D+a}$; therefore $x+a=\frac{\frac{1}{2}Da}{\frac{1}{2}D+a}+a=\frac{Da+a^2}{\frac{1}{2}D+a}$; that is, $\frac{1}{2}D+a : D+a :: a : x+a$; or, CT : YT :: FT : PT. *Q. E. D.*

Proposition XLV.

As the Semi-diameter added to the external Part, is to the Semi-diameter; so is the Diameter added to the external Part, to the Diameter less by the Abscissa; that is, CT : CF :: YT : YP.

Demonstration.

By the 41st, $\frac{1}{2}D-x : \frac{D}{2} :: D-x : D+a$, and (by Prop. 39) $\frac{1}{2}D-x : \frac{D}{2} :: \frac{D}{2} : \frac{1}{2}D+a$; therefore $\frac{1}{2}D+a : \frac{D}{2} :: D+a : D-x$; or, CT : CF :: YT : YP. *Q. E. D.*

Proposition XLVI.

As the Diameter less by the Abscissa, is to the Semi-Diameter; so is the Subtangent, to the external Part of the Diameter produced to the Tangent; that is, YP : CF :: PT : FT.

Demonstration.

By the 40th, $\frac{1}{2}Da-ax=\frac{1}{2}Dx$; therefore $\frac{1}{2}D-x=\frac{Dx}{2a}$, $D-x=\frac{D}{2}+\frac{Dx}{2a}=\frac{Da+Dx}{2a}$; that $D-x : \frac{1}{2}D :: a+x : a$; or, YP : CF :: PT : FT. *Q. E. D.*

PRO-

PROPOSITION XLVII.

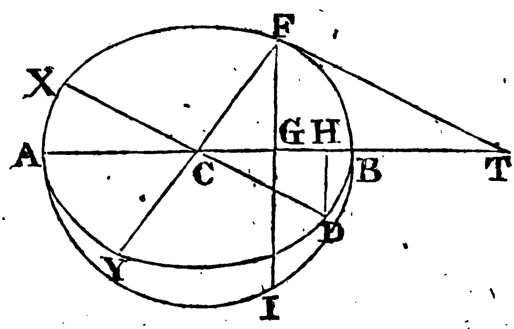
As the Diameter added to the external Part, is to the Semi-diameter; so is the Subtangent, to the Abscissa; that is, YT : CF :: PT : PF.

DEMONSTRATION.

By the 40th, $\frac{1}{2}Da = \frac{1}{2}Dx + ax$; therefore $\frac{1}{2}D + a = \frac{\frac{1}{2}Da}{x}$, and $D + a = \frac{1}{2}D + \frac{\frac{1}{2}Da}{x} = \frac{\frac{1}{2}Dx + \frac{1}{2}Da}{x}$; that is, $D + a : \frac{1}{2}D :: x + a : x$; or, YT : CF :: PT : PE. *Q. E. D.*

PROPOSITION XLVIII.

If, from the Extremities of two conjugate Diameters, Ordinates be drawn to the Axe; then the Distance on the Axe between the Center and one of these Ordinates, is a mean Proportional between the Segments of the Axe made by the other Ordinate; that is, AG : CH :: CH : GB; or, AH : CG :: CG : HB.



2D-

DEMONSTRATION.

Draw the Tangent FT which will be parallel to CD (by the 34.) And let BC be put $=t$, CH$=a$, GT$=s$, and CG$=x$; then GB$=t-x$ and AG$=t+x$; whence (by Eu. 4 & 22. 6.) $\overline{GT}^2 : \overline{CH}^2 :: \overline{FG}^2 : \overline{DH}^2$ :: (by Prop. 1.) AG$\times$GB : AH$\times$HB : But (by Eu. 5. 2.) AG$\times$GB$=\overline{BC}^2-\overline{CG}^2$, and AH$\times$HB$=\overline{BC}^2-\overline{CH}^2$; threfore $\overline{GT}^2 : \overline{CH}^2 :: \overline{CB}^2 - \overline{CG}^2 : \overline{CB}^2 - \overline{CH}^2$; that is, $s^2 : a^2 :: t^2 - x^2 : t^2 - a^2$; but (by the 13.) CG : GB :: AG : GT; that is, $x : t-x :: t+x : s = \frac{t^2-x^2}{x}$; therefore $s^2 = \frac{\overline{t^2-x^2} \times \overline{t^2-x^2}}{x^2} : t^2 - x^2 :: a^2 : t^2 - a^2$; or $\frac{t^2-x^2}{x^2} : 1 :: a^2 : t^2 - a^2$; whence, by Composition, $t^2 : a^2 :: \frac{t^2}{x^2} : \frac{t^2-x^2}{x^2}$ and by multiplying the third and fourth Term by x^2, $t^2 : a^2 :: t^2 : t^2 - x^2$; therefore $a^2 = t^2 - x^2$ and $t+x : a :: a : t-x$; or, AG : CH :: CH : CB : In like manner it may be proved, that AH : CG :: CG : HB.

COROLLARY 1.

Hence it will be no difficult Matter, to draw a conjugate Diameter, without drawing a Tangent: For let the Ordinate FG be produced to meet the Periphery of a Circle described on the transverse Axe in I; make CH$=$GI, and from H draw the Ordinate HD; then from the Point D, through the Center draw DCX and it will be the conjugated Diameter required.

Co-

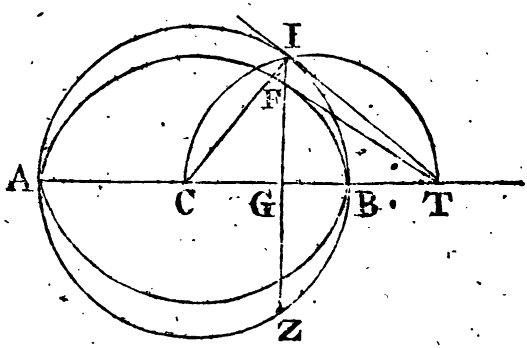
COROLLARY 2.

The Sum of the Squares of any two Diameters, as DX and FY, is equal to the Sum of the Squares of the transverse and conjugate Axes.

For if a be put for CG then (by Prop. 1.) $t^2 : c^2 :: (AH \times HB =$ by this Prop. $\overline{CG}^2 =) \ a^2 : \overline{HD}^2 = \frac{c^2 a^2}{t^2}$, and (by this Prop.) $\overline{CH}^2 = AG \times GB = \frac{1}{4}t^2 - a^2$; therefore (by Eu. 47. 1.) $CD^2 = \frac{1}{4}t^2 - a^2 + \frac{c^2 a^2}{t^2}$; also (by Prop. 1.) $t^2 : c^2 :: (AG \times GB =) \ \frac{1}{4}t^2 - a^2 : \overline{GF}^2 = \frac{1}{4}c^2 - \frac{a^2 c^2}{t^2}$; therefore $\overline{CF}^2 = a^2 + \frac{1}{4}c^2 - \frac{a^2 c^2}{t^2}$; whence $\overline{CD}^2 + \overline{CF}^2 = \frac{1}{4}t^2 + \frac{1}{4}c^2$.

PROPOSITION XLIX.

If any Ordinate to the Axe, as GF, be produced to meet the Periphery of a Circle described on the transverse Axe in I, and from the Points F and I, Tangents be drawn to the respective Curves, they will both intersect the Axe produced in one and the same Point T.



DEMONSTRATION.

Draw the Radius CI, and put $BG=x$, $AB=t$, $BT=a$, and $GI=y$; then, $TG\times GC=\overline{GI}^2=BG\times AG$; that is, $\overline{a+x}\times\overline{\tfrac{1}{2}t-x}=y^2=tx-x^2$; therefore $\tfrac{1}{2}tx=\tfrac{1}{2}ta-ax$; or $\tfrac{1}{2}t-x:\tfrac{1}{2}t::x:a$; but in the Ellipse (by the 15.) $\tfrac{1}{2}t-x:\tfrac{1}{2}t::x:a$: In both Curves the three first Terms are the same; therefore the fourth Term *viz.* $a=BT$ must be the same, and consequently, the Point T, is that wherein both Tangents will intersect. *Q. E. D.*

COROLLARY 1.

Hence any Point in the Curve being given, we have an easy Method of drawing a Tangent to touch that Point: For if from the given Point F, the Ordinate FG be drawn, and produced to meet the Periphery of the circumscribing Circle in the Point I, and a Tangent IT be drawn touching the Circle in that Point; then where that Tangent cuts the Axe produced, as in T, is the Point, to which from the given Point (*viz.* F) in the Ellipse a right Line be drawn, it will be a Tangent.

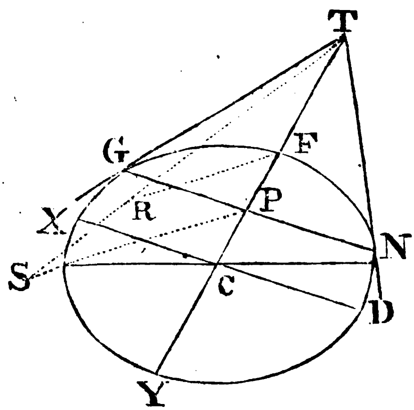
COROLLARY 2.

Hence also, if from a Point T given in the Axe produced, it be required to draw a Tangent to the Ellipse, it is easily done: For if on CT be described the Semicircle CIT cutting the Periphery of the circumscribing Circle in I; and BZ be made equal to BI, and IZ be drawn; then from the Point F where that Line cuts the Curve, draw the right Line TF and it will touch the Curve of the Ellipse in the Point F.

Scholium. From this Propoſition it is evident, that all the Properties of the Tangents which have been demonſtrated in the Ellipſe from Prop. 13, to Prop. 21, incluſively, hold good alſo in the Circle.

PROPOSITION L. *Problem.*

From any given Point (as T) any where without the Ellipſe to draw a Tangent.



CONSTRUCTION.

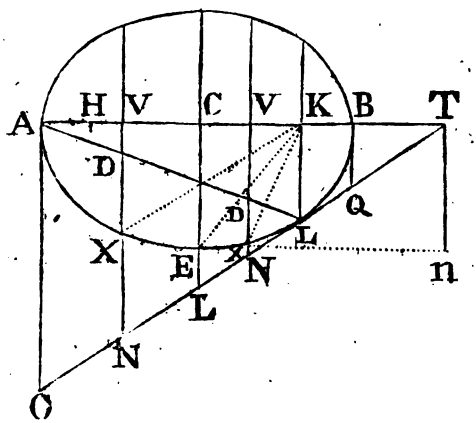
From the given Point T through the Center C draw the right Line TFCY, and to the Diameter FY (by Corol. to Prop. 48.) draw the conjugate Diameter DX; make the Angle YTS at pleaſure, and in TS take TR=TC and SR=CF; join RF, and draw SP parallel thereto; laſtly, thro' P and parallel to the conjugate Diameter draw GN; then if from the Point T, to G or N right Lines be drawn, they will touch the Ellipſe in thoſe Points.

DEMONSTRATION.

By Construction and Eu. 26. 2. TR : RS :: TF : FP; but TR=TC and RS=CF; therefore TC : CF :: FT : FP, and (by Prop. 43.) TG or TN are Tangents.

PROPOSITION LI.

If any Ordinate to the Axe (as VX) be continued to a Point (N,) in the focal Tangent (TO;) then the Distance (VN) from the Axe to that Point in the Tangent, will be equal to (KX) the Distance from the Focus to the Extremity of that Ordinate.



DEMONSTRATION.

Put CK=b, BC=c, CV=d; then AK=$b+c$, BK=$c-b$, VK=$b\pm d$, BV=$c\pm d$, and AV=$c\pm d$, and K being the Focus, (by the 4.) KL will be half the Parameter of the Axe: and (by the 3d.) CB : AK :: KB : KL; or, $c : c+b :: c-b : \frac{c^2-b^2}{c}$=KL=$\frac{1}{2}p$;

also,

also CK : CB :: CB : CT; or $b : c :: c : \frac{c^2}{b} = CT$, by the 14; but CT—CK=KT; that is $\frac{c^2}{b} - b = \frac{c^2 - b^2}{b} = KT$, also CT±CV=VT; that is $\frac{c^2}{b} \pm d = \frac{c^2 \pm bd}{b} = VT$; but by similar Triangles KT : KL :: VT : VN; or $\frac{c^2 - b^2}{b} : \frac{c^2 - b^2}{c} :: \frac{c^2 \pm bd}{b} : \frac{c^2 \pm bd}{c} = VN$.

Again, by the 2d, CB : KL :: AV×BV : $\overline{VX}^2$; or $c : \frac{c^2 - b^2}{c} :: c^2 - d^2 : \frac{c^4 - b^2c^2 - c^2d^2 + d^2b^2}{c^2} = \overline{VX}^2$; and $\overline{VK}^2 = b^2 \pm 2bd + d^2$; but $\overline{VK}^2 + \overline{VX}^2 = \overline{KX}^2$; that is, $\frac{c^4 + 2bdc^2 + d^2b^2}{c^2} = \overline{KX}^2$, and (by extracting the Square-Root) $\frac{c^2 \pm bd}{c} = KX =$ by what is already proved in the first Part, VN. *Q. E. D.*

To be continued.

ANSWERS

ANSWERS

TO THE

PROBLEMS

Proposed in the Second NUMBER.

PROBLEM I. *Answered by* Sam. Farrer *of* London.

THE present Value of an Annuity of one Pound for 31 Years, Interest at 5 *l. per Cent.* being 15.592811, for eight Years 6.463213, and for 10 Years 7.721735; it is manifest that the Excess 9.129598, by which the first of these exceeds the second, will be to 7.871076, that by which it exceeds the third, as 400 the present Value of the former Reversion, to 344.8596, or 344 *l.* 17 *s.* 2 *d*$\frac{1}{4}$. the present Value of the latter.

Otherwise, by John Turner *of* London.

The Reversion of an Annuity at 5 *l. per Cent.* for 23 Years, after 8 Years, is 9.130; and that for 21 Years, after 10, is 7.872; therefore 9.130 : 400 :: 7.872 : 344.8849, or 344 *l.* 17 *s.* 8 *d*$\frac{1}{4}$.

PRO-

PROBLEM II. *Anſwered by* John Turner *of* London.

Let the two Lights be repreſented by A, the three Lights by B, and let the Quantity of Light received from any one of the two Lights be ſuppoſed $=1$; then $3:4::2:\frac{8}{3}$ the Quantity of Light received from B, therefore $\frac{8}{9}$ will be the Quantity of Light received from one of the Lights in the firſt Poſition. Alſo $5:3::2:\frac{6}{5}$ the Quantity of Light received from B in the ſecond Poſition, therefore $\frac{2}{5}$ will be the Quantity of Light received from one of the Lights in that Poſition.

Now, the Quantity of Light received from luminous Bodies of equal Magnitudes, being in the reciprocal duplicate Ratio of their Diſtances, we ſhall have $\sqrt{1}:\sqrt{\frac{8}{9}}::$ Diſtance from B : Diſtance from A; therefore $\frac{\sqrt{8}}{3+\sqrt{8}}=$ the Diſtance from A at the firſt Obſervation. Again $\sqrt{1}:\sqrt{\frac{2}{5}}::$ Diſtance from B : Diſtance from A; whence $\frac{\sqrt{2}}{\sqrt{5}+\sqrt{2}}=$ the Diſtance from A at the ſecond Obſervation.

But $\frac{\sqrt{8}}{3+\sqrt{8}}-\frac{\sqrt{2}}{\sqrt{5}+\sqrt{2}}=.09786$, therefore $.09786:400::1:4087.4719$ the whole Diſtance of the Lights; whence 2104.9254 the Diſtance from B, and 1983.5683 the Diſtance from A at the firſt Obſervation.

Other-

Otherwise, by Samuel Farrer.

The Number in one Set of Lights, being to the Number in the other, as 2 to 3; the Quantity of Light emitted from them at the first Obſervation, in the Ratio of 3 to 4; and at the latter Obſervation in the Ratio of 5 to 3 reſpectively; it follows that the Quantity of Light received from any one in the former Sett, will be to that received from any one in the latter, at the firſt Obſervation, in the given Ratio of $\frac{3}{2}$ to $\frac{4}{3}$, and at the other Obſervation in the given Ratio of $\frac{5}{2}$ to 1, becauſe all the Lights are equal among themſelves by Hypotheſis: Hence if the Diſtance between the Traveller and the firſt Set of Lights, at the firſt Obſervation, be repreſented by x, it will be $\sqrt{\frac{4}{3}} : \sqrt{\frac{3}{2}} :: x \frac{3x}{2\sqrt{2}}$ the Diſtance betwixt him at that Time, and the other Set; becauſe the Quantity of Light emitted by luminous Bodies are inverſely as the Squares of the Diſtances; but by proceeding forwards 400 Yards the Diſtances, from the Lights, do then become $x - 400$ and $\frac{3x}{2\sqrt{2}} + 400$ reſpectively; therefore $\sqrt{\frac{5}{2}} : \sqrt{1} :: \frac{3x}{2\sqrt{2}} + 400 : x - 400$, from whence $x = \frac{800 \times \sqrt{5 + \sqrt{2}}}{2\sqrt{5} - 3} = 1983.6654$, &c. is given, and conſequently the other Diſtances, from what has gone before, may now be eaſily determined.

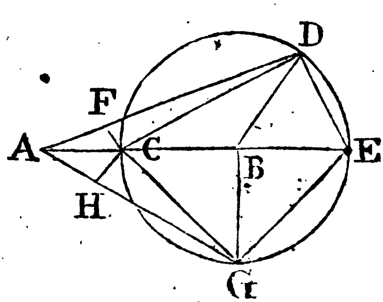
PROBLEM III. *Anſwered by* John Turner.

It is demonſtrated by all writers of Trigonometry, that the Sum of the Sides, including any given Angle

Angle of a plain Triangle, is to their Difference, as the Tangent of half the Sum of the unknown Angles, is to the Tangent of half their Difference; therefore, if the including Sides of two different Triangles be respectively equal, it follows, by Equality, that as the Tangent of half the Sum of the unknown Angles in one Triangle, is to the Tangent of half their Difference, so is the Tangent of half the Sum of the unknown Angles in the other Triangle, to the Tangent of half their Difference. But if the included Angle of one Triangle be a right one, half the Sum of the other two will be half right, and half their Difference the Excess of the greater above half a right one; whence the Truth of the Theorem is manifest.

Otherwise, by Samuel Farrer.

Let ABD be the proposed Triangle, and upon the Center B, with the Interval BD, let a Circle DEGC be describ'd intersecting AB, produced, in C and E: Also let BG be drawn perpendicular to ABE, and draw CD, CG, DE, and GE, and let CF and CH be parallel to ED and GE respectively. It is evident that ECD ($\frac{1}{2}$EBD) is is equal to half the Sum of the Angles, BDA+BAD; and that CDF$=\frac{1}{2}$ their Difference, because BCD=BDC. It

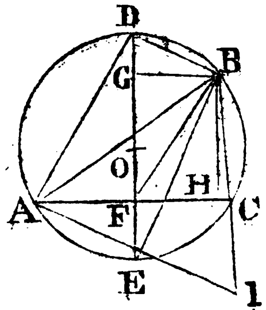


F is

is likewise evident that the Angles EDC, DCF, EGC, and GCH, are all right Angles; but BG (BD) : BA :: as Radius : Tangent of BGA; whence it appears that BGA is the Angle found, or understood in the first Part of the Theorem, whose Excess above half a right Angle (BGC) is the Angle CGH. Now, by similar Triangles, EG (GC) : CH (:: AE : AC) :: ED : CF; but GC : CH :: Radius : Tangent CGH; and ED : CF :: Tangent of ECD : Tangent CDF, therefore by Equality Radius : Tangent of CGH :: Tangent ECD : Tangent CDF. *Q. E. D.*

PROBLEM XXIV. *Answered by* John Turner.

Suppose the Triangle ABC, which is inscribed in the Circle ADCE, to be similar to the required one. Let s and c represent the Sine and Cosine of half the Difference of the Angles at the Base, putting the Radius of the Circle $=1$, and FO$=x$; then Ra-



dius (1) : DE (2) :: S. EDB (c) : BE$=2c$, and $1 : 2 :: s : 2s=$DB; also $1 : 2c :: c : 2c^2=$GE, and $1 : 2c :: s : 2cs=$GB. But GE—EF$=2c^2-\overline{1+x}$ (GF)$=$ BH, and by the Property of the Circle $\overline{1+x}\times\overline{1-x}$ $=$AF^2, therefore AD$=\sqrt{2+2x}$.

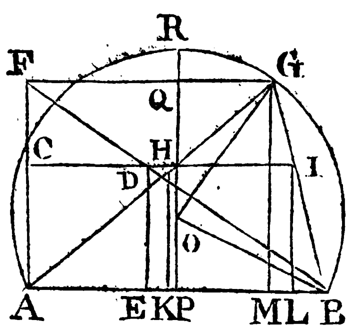
Now

Now if BC be produced to I, so that BI=BA, annd A, I be joined, the Triangles ABD, AIC, will be similar; therefore $\sqrt{2+2x} : 2s :: 2\sqrt{1-x^2} : \frac{4s\sqrt{1-x^2}}{\sqrt{2+2x}}$ = CI. Moreover $\sqrt{BH^2+FH^2}$ = FB = $\sqrt{4c^2x+1-2x+x^2}$; whence by Similarity of Figures $\sqrt{4c^2x+1-2x+x^2} : \frac{4s\sqrt{1-x^2}}{\sqrt{2+2x}} :: a$ (the given bisecting Line) : d (the given Difference of the Sides; consequently $d\sqrt{4c^2x+1-2x+x^2} = \frac{4as\sqrt{1-x^2}}{\sqrt{2+2x}}$, and $d^2x^2 \left.\begin{matrix} +4c^2d^2 \\ -2d^2 \\ +8a^2s^2 \end{matrix}\right\} x = 8a^2s^2 - d^2$.

PROBLEM XXV. *Answered by* John Turner.

CONSTRUCTION.

Upon the given Base AB describe a Segment of a Circle to contain the given Angle, and let ACDE be the given Square; draw BDF meeting AC produced in F, and draw FG, parallel to AB, intersecting the Circle in G; join A, G, and B, G, then will ABG be the Triangle required.



DEMONSTRATION.

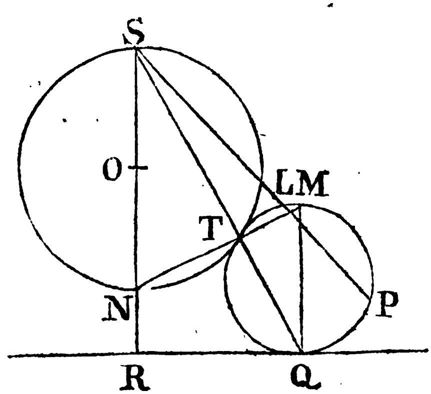
Make GM perpendicular to AB, and let CD be produced cutting the Sides of the Triangle in H and I; then by similar Triangles FC : CD :: (FA : AB :: GM : AB) GN : HI, therefore GN being equal to FC, HI will be equal to CD=DE=HK, which was to be proved.

NUMERICALLY.

As BE : ED :: AB : AF=GM the perpendicular Height of the Triangle; and, as S.POB : Rad. :: PB : BO, alſo S,POB : S,PBO :: PB : PO, whence QO is given =GM—OP, and conſequently $QG = \sqrt{OG^2 - OQ^2}$; but OG : QG :: Rad. : S,QOG the Difference of the Angles at the Baſe; from whence all the Angles and Sides may be found.

PROBLEM XXVI. *Anſwered by* B. Oxon.

Let P be the given Point, QR the given right Line, and STN the given Circle; then thro' O the Center of the given Circle, draw the right Line RS perpendicular to QR and join S, P; take SL to SN as SR to SP, and thro' the Points P, L, deſcribe, *by Prob.* 39. *Page* 185. *Simpſon's Geometry*, the Circle PLTQ to touch the right Line QR, *and the Thing is done.*



DE-

DEMONSTRATION.

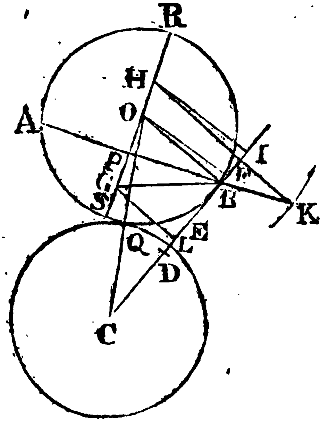
Draw QS cutting the Circle STN in T, and join N, T; because the Angles T and R are right Angles, and the Angle S common, the Triangles SNT and SQR are similar, therefore SQ : SN :: SR : TS, and $SN \times SR = ST \times SQ$; but $SP \times SL = SN \times SR$ by Construction, therefore $SP \times SL = ST \times SQ$, whence it is evident that the Points P, L, T, Q are in a Circle; but the Point T is in the Circle SNT, therefore the Circles either cut, or touch each other in that Point; let now QM be drawn, from the Point of Contact Q, perpendicular to QR, and join T, M, then because MTQ is a right Line and vertical to STN, it is evident that the Lines NT and TM are one continued right Line, therefore the Circles touch each other in the Point T. *Q. E. D.*

PROBLEM XXVII. *Answered by* John Turner, *from Page* 160, *Simpson's Geometry.*

CONSTRUCTION.

Join A, B, the two given Points, also draw BC from one of those Points to the Center of the given Circle; bisect AB with the indefinite Perpendicular RPS; also bisect BC, and, from the Point of Bisection D, take DE a third Proportional to 2BC and CL, then take EI=CL, and draw EG and IH, perpendicular to BC, meeting RS in G and H; join G, B, and from the Center H, with the Interval BC, let an Arch be described cutting PB, produced (if need be) in K; draw KH and BO parallel thereto, meeting RS in O; then upon O as a Center, with the Interval OB, let a Circle be described, *and the Thing is done.*

DE-



Demonstration.

Let OF be perpendicular to CB: Then, because of the parallel Lines, KH (BC) : BO (: : GH : GO) : : EI (CL : EF, and consequently $BC \times EF = BO \times CL$. Moreover, by Construction, $2BC : CL :: CL : DE$; therefore $BC \times DE = \frac{1}{2}CL^2$; to which adding $CB \times EF = BO \times CL$, we have $BC \times DF = BO \times CL + \frac{1}{2}CL^2$, and therefore $2BC \times DF = 2BO \times CL + CL^2$: But $2BC \times DF$ is also equal $CO^2 - BO^2$ (*by Cor. to Theor.* 8. *Page* 37. *Simpson's Geometry*;) therefore $2BO \times CL + CL^2 = CO^2 - BO^2$, or $BO^2 + 2BO \times CL + CL^2 = CO^2$, that is, $\overline{BO + CL}^2 = CO^2$; whence $CO - CQ (= CL) = BO$, therefore it is evident that the Circle O touches the given Circle in Q and passes through B and A, because RS is perpendicular to, and bisects, AB in P. *Q. E. D.*

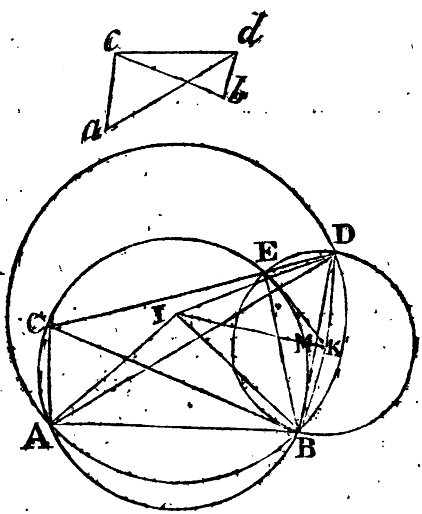
Problem XXVIII. *Answered by the Proposer.*

Construction.

Suppose AB to be the given Side, and *acb*, *bcd*, *cda* and *bda* the given Angles.

 Upon

Upon the given Side AB let two Segments of Circles be deſcribed, one of them ACB to contain an Angle equal to *acb*, and the other ADB an Angle equal to *adb*; make ABE equal to the Supplement of *acd* to 2 right Angles, and upon BE deſcribe a Segment of a Circle, to contain an Angle equal to *cdb*, cutting the Circle ADB in D; then thro' E draw DEC, and join A, C, and B, D, and ABDC will be the Trapezium required.



DEMONSTRATION.

Draw BC and AD.

Since the Angle ACD+ABE is equal to 2 right Angles, and the Angle ABE is the Supplement of *acd* to 2 right Angles, by Conſtruction, therefore is ACD=*acd*, and conſequently BCD=*bcd*: alſo, becauſe the Angle EDB is equal to *cdb*, and the Angle ADB=*adb*, by Conſtruction, therefore is ADC =*adc*. *Q. E. D.*

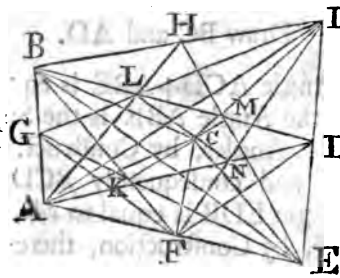
CAL-

Calculation.

From the Center I, of the Circle ABD, to the Points A, B, D, draw the Radii IA, IB, ID; and from the Center K, of the Circle BED, draw the Radii KB, KE, and KD; also join A, E, and I, K: Then in the Triangle ABE are given all the Angles and the Side AB, whence BE and likewise BK will be given; but the Angle EBI (=ABE—ABI) is given, therefore the Angle KBI is also given; from whence, because the Sides BI and BK are given, the Angle BIK=DIK will be found: Again, in the right-angled Triangle BMI are given all the Angles and the Side BI, whence BM=MD is given, $=\frac{1}{2}$BD one of the required Sides of the Trapezium; and from thence the rest may easily be determined.

Problem XXIX. *Answered by* Samuel Farrer.

Let ABDE represent the Trapezium; BF, EG, AH, AI, and DF, DG, EH, BI, Lines drawn from the several Angles bisecting the Sides AE, AB, BD, and DE, two and two in the Order in which they are written: Then if from the Intersection K of the Lines BF, EG, to the Intersection M of the Lines BI, HE, be drawn the Line KM; and from the Intersection L of the Lines AH, DG, to the In-



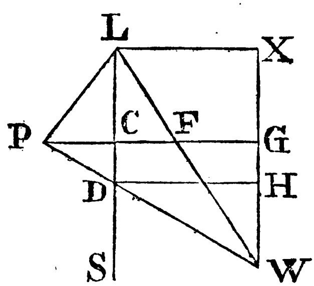
tersection

tersection N of the Lines AI, DF, be drawn the Line LN, the latter will intersect the former in C the Center of Gravity of the Trapezium. For since the Intersections L, N, are the Centers of Gravity of the two Triangles made by one of the Diagonals AD; the Intersections K, M, the Centers of Gravity of the two Triangles made by the other Diagonal BE; it is manifest, that the Center of Gravity of the Trapezium will be somewhere in the Line LN, and also somewhere in the Line KM; therefore it can only be at C, the Intersection of these Lines; from whence and the Principles of Mechanics it follows, that $KC \times$ by the Triangle $BAE = CM \times$ by the Triangle BDE, and $LC \times$ by the Triangle $ABD = CN \times$ by the Triangle AED. In order to determine which, draw the Lines CF, FG, and FI, and assume KC and CN, both unknown; then in the Triangle BCF will be given the two Sides BC, CF, the Line KC, and the Ratio of the Segments BK, KF, as 2 to 1; whence BF, and the Sines and Cosines of all the Angles may separately be determined, in Terms of the assumed Line KC, and given Quantities; because $BC^2 \times KF + CF^2 \times BK = \overline{KC^2 + BK \times FK} \times BF$: But the Sine and Cosine, of the Angle AFC being given, and those of BFC determined; the Sine and Cosine of their Difference AFB may also be determined; then in the Triangle BFA will be given the two Sides and the Angle included; whence BF, the Sines and Cosines of the other two Angles, and consequently GF may be said to be given; but two third Parts of GF, is equal to the Line LN, from whence taking the assumed Line CN, the Line LC becomes known. Now the two Sides AB, AE, together with their included Angle, being given; the Area of the Triangle ABE from thence becomes also known. After the same Method of Reasoning CM, and the Area of the Triangle AED will become known; but the Sines and Cosines of the Angles BCF, DCF, being

given, the Sine and Cosine of the Angle BCD from thence may be inferred; then in the Triangle BCD will be given, the two Sides BC, CD, and their included Angle; whence BD and the Sines and Cosines of the other two Angles may be determined. Moreover all the Sides of the Triangles ABC, DCE, being given, the Sines and Cosines of all the Angles, may from thence be determined; and by an easy Consequence, from what has gone before, the Sines of the Angles ABD, BDE; then in each of the Triangles ABD, BDE, will be given, two Sides and an Angle included, whence the Areas become known. *Q. E. I.*

QUEST. 30. *Answered by* Samuel Farrer.

Let LP (=20 Parts) make with LS, the Meridian departed from, an Angle of 45°, and let PD be so drawn between the South and the East, that,



when produced 25 Parts further towards W, a right Line drawn from thence to L, the Lizard, may make an Angle with LS of 56°. 15′; also let WX be parallel to LS, and PG, LX, perpendicular thereto; then will PD represent the Distance sailed in Company with the Privateer, W the Place when speaking to the Man of War, WL the Distance from the Lizard, WX the Difference of Latitude, and CG or LX the Departure from the Meridian. Now in the Triangle PLF, are given all the Angles

and

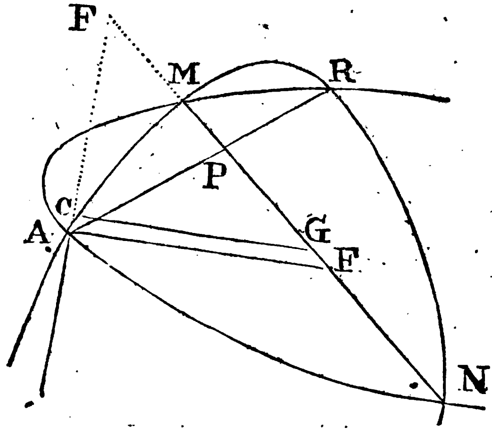
and the Side PL; whence CF and PF may be said to be given; for the former of which put a, and for the latter b; also put DW$=c$, Cotangent of the Angle CLF$=t$, and FG$=x$; then will CG$=$DH$=a+x$, and PG$=b+x$; therefore, *by Trig.* $1 : t :: x : tx=$GW; whence PW$^2=\overline{b+x}^2+t^2x^2$; but the Triangles PWG, DWH, being equiangular, $\overline{b+x}^2+t^2x^2 : c^2 :: \overline{b+x}^2 : \overline{a+x}^2$, therefore $\overline{\overline{b+x}^2+t^2x^2}\times\overline{a+x}^2=c^2\times\overline{b+x}^2$; from whence the Value of x may be determined, and consequently all the other Lines in the Figure, the Signification of which are expressed above.

PROBLEM XXXI. *Answered by* John Turner.

Put $y=\frac{1}{2}$ the Head Diameter, $x=\frac{1}{2}$ the Bung Diameter, and $a=$ the given Diagonal; then $\sqrt{a^2-\overline{x+y}^2}=\frac{1}{2}$ the Length, and consequently the Content of the Cask, will be as $\overline{8x^2+4y^2}\times\sqrt{a^2-x^2-2xy-y^2}$. Now it is manifest that this Expression will be a Maximum when $y=0$, therefore the required Content of the greatest Cask will be as $8x^2\times\sqrt{a^2-x^2}$; whence $a^2x^4-x^6$, which is as the Square thereof, must be a Maximum, and consequently $4a^2x^3=6x^5$, and $x=a\sqrt{\frac{2}{3}}$; therefore it appears that the required Cask must be the whole Spheroid whose Axis is $2a\sqrt{\frac{1}{3}}$, and whose conjugate Diameter is $2a\sqrt{\frac{2}{3}}$.

PROBLEM XXXII. *Answered by* Samuel Farrer.

Let AMNR be the four given Points thro' which the Curve of the Parabola is to be described, and let the right Lines AR, MN, be drawn intersecting each other in P; take AP$\times$PR, to MP$\times$NP, as AP^2, to PF^2, and draw the Line AF; then thro' G, the Middle of MN, draw GC parallel to AF, and make



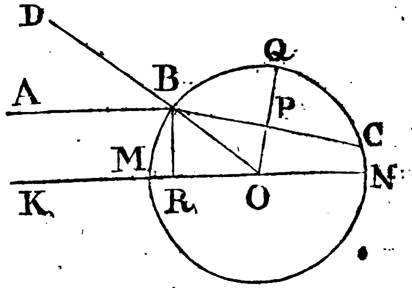
MF×FN, to MG×GN (GM^2,) as AF to FG; and a Parabola described thro' the Vertex C of the Diameter CG, whose Parameter is a Third-proportional to CG, and GM, and Ordinates parallel to MN; will be that required: This is demonstrated by the Writers on Conic Sections.

It is to be observed, from the first Proportion of the Construction, that two different Parabolas may be described thro' the four given Points; but that having the least Parameter when referred to the Axis solves the Problem.

PROBLEM XXXIII. *Answered* a Fratre Euclidis.

From the Center O draw OBD, and OPQ, perpendicular to BC, cutting the Circle in Q, and let BR the Sine of Incidence BOM, or ABD be denoted by x, and the Radius OB by Unity: Then the Sine of Incidence being to the Sine of Refraction in a constant Ratio; suppose that of m to n, it will be as $m : n :: x : \frac{nx}{m}$ = OP the Sine of Refraction OBP. Now the Arch NC being a Maximum, by the Question, its Supplement MBQC must be a Minimum, and the Fluxion thereof, or that of its Equal MB+ 2BQ,

2BQ, equal to nothing. But the Fluxion of any Arch, Radius being Unity, is known to be equal to the Fluxion of the Sine divided by its Cosine; therefore the Sine and Cosine of the Arch MB being ex-



pressed by x and $\sqrt{1-x^2}$, and those of BQ by $\sqrt{\frac{m^2-n^2x^2}{m^2}}$ and $\frac{nx}{m}$ respectively; the Fluxions of those Arches MB and BQ will be expressed by $\frac{\dot{x}}{\sqrt{1-x^2}}$ and $\frac{n\dot{x}}{\sqrt{m^2-n^2x^2}}$, and so $\frac{\dot{x}}{\sqrt{1-x^2}}-\frac{2n\dot{x}}{\sqrt{m^2-n^2x^2}}=0$; which Equation, by transposing, dividing by $\dot{x}$, and squaring both Sides becomes $\frac{1}{1-x^2}=\frac{4n^2}{m^2-n^2x^2}$, and this, solved, gives $x=\sqrt{\frac{4n^2-m^2}{3n^2}}$ for the Distance of the Line AB from the Line KMN, or Sine of Incidence required.

But it may be proper to observe that if m be greater than $4n$, the Value of x will become impossible, in which Case all the Rays will emerge on the contrary Side of the Axis, and that will be the lowest of all whose Incidence is 90 Degrees.

PRO-

PROBLEM XXXIV. *Answered by* Samuel Farrer.

Let $p=0.7854$, &c. $b=$ one of the Diameters, $b=$ the other, $y=b-b$, their Difference, and $x=$ the Length; also let n be the fractional Number or Quantity, by which if the Difference y be multiplied, and their Product taken from b the greater Diameter; the Remainder may be the Diameter of a Cylinder equal in Magnitude, and of the same Length as the Frustum. Then will $\overline{b-my}^2\times px$ be the Content of such a Frustum, whose Fluxion $=p\dot{x}\times\overline{b-my}^2-2pm\dot{y}x\times\overline{b-my}$; but the Fluxion of this or any other Solid is $=p\dot{x}\times\overline{b-y}^2$ universally; therefore $\overline{b-y}^2\times\dot{x}=\dot{x}\times\overline{b-my}^2-2mx\dot{y}\times\overline{b-my}$; whence $\frac{\dot{x}}{x}=$

$$\frac{m\dot{y}}{\overline{1-m}\times y}+\frac{\overline{m-m^2}\times\dot{y}}{2b-2bm+\overline{m^2-1}\times y}=\frac{m}{1-m}\times\frac{\dot{y}}{y}+$$

$$\frac{m-m^2}{m^2-1}\times\frac{\dot{y}}{\frac{2b-2bm}{m^2-1}+y}=\frac{m}{1-m}\times\frac{\dot{y}}{y}-\frac{m}{m+1}\times$$

$$\frac{\dot{y}}{-\frac{2b}{m+1}+y}.$$

Now the Fluent of $\frac{\dot{x}}{x}$ is $=$ the Hyperbolical Logarithm of x, that of $\frac{\dot{y}}{y}=$ the Hyperbolical Logarithm of y, and that of $\frac{\dot{y}}{\frac{-2b}{m+1}+y}=$ the Hyperbolical Logarithm of $y-\frac{2b}{m+1}$; therefore the

the Log of $x = \frac{m}{1-m}$ L. $y - \frac{m}{m+1}$ L. $y - \frac{2b}{m+1}$;

or $x = \dfrac{y^{\frac{m}{1-m}}}{\overline{y-\frac{2b}{m+1}}\Big|^{\frac{m}{m+1}}}$. *Q. E. I.*

PROBLEM XXXV. *Answered by* Samuel Farrer.

Let $\overline{a+cx^n}|^m$ be reduced to a Series, and it will become $= a^m + mca^{m-1}x^n + m.\frac{m-1}{2}c^2a^{m-2}x^{2n} + m.\frac{m-1}{2}.\frac{m-2}{3}c^3a^{m-3}x^{3n} +$ &c. which multiplied by $d\dot{x}x^{pn-1}$ gives $da^m\dot{x}x^{pn-1} + dmca^{m-1}x^{pn+n-1}\dot{x} + dm.\frac{m-1}{2}c^2a^{m-2}x^{pn+2n-1}\dot{x} + dm.\frac{m-1}{2}.\frac{m-2}{3}.c^3a^{m-3}x^{pn+3n-1}\dot{x} +$ &c. for the Fluxion of the Area, whose Fluent $= \frac{dx^{pn}}{n}$ into $\frac{a^m}{p} + \frac{mca^{m-1}x^n}{p+1} + \frac{m}{1}.\frac{m-1}{2}.\frac{a^{m-2}c^2x^{2n}}{p+2} + \frac{m}{1}.\frac{m-1}{2}.\frac{m-2}{3}.\frac{a^{m-3}c^3x^{3n}}{p+3}$, &c. or $\frac{dx^{pn}}{n}$ into $\frac{\overline{v-cx^n}|^m}{p} + m.\frac{\overline{v-cx^n}|^{m-1}cx^n}{p+1} + m.\frac{m-1}{2}\frac{\overline{v-cx^n}|^{m-2}c^2x^{2n}}{p+2} + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{\overline{v-cx^n}|^{m-3}c^3x^{3n}}{p+3} +$ &c. because $a = v - cx^n$ by Hypothesis; but $\frac{\overline{v-cx^n}|^m}{p} = \frac{v^m}{p} -$

and

$\frac{mv^{m-1}cx^n}{p} + m.\frac{m-1}{2}.\frac{v^{m-2}c^2x^{2n}}{p} + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{v^{m-3}c^3x^{3n}}{p} +$ &c. $m.\frac{\overline{v-cx^n}|^{m-1}cx^n}{p+1} = \frac{mv^{m-1}cx^n}{p+1} - m.\frac{m-1}{2}.\frac{v^{m-2}c^2x^{2n}}{p+1} + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{v^{m-3}c^3x^{3n}}{p+1}$ &c. $m.\frac{m-1}{2}.\frac{\overline{v-cx^n}|^{m-2}}{p+2}\times c^2x^{2n} = m.\frac{m-1}{2}.\frac{v^{m-2}c^2x^{2n}}{p+2} - m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{v^{m-3}c^3x^{3n}}{p+2} +$ &c. and $m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{\overline{v-cx^n}|^{m-3}c^3x^{3n}}{p+3} = m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{v^{m-3}c^3x^{3n}}{p+3}$, &c. therefore $\frac{\overline{v-cx^n}|^m}{p} + m.\frac{\overline{v-cx^n}|^{m-1}cx^n}{p+1} + m.\frac{m-1}{2}.\frac{\overline{v-cx^n}|^{m-2}c^2x^{2n}}{p+2} + m.\frac{m-1}{2}.\frac{m-2}{3}.\frac{\overline{v-cx^n}|^{m-3}c^3x^{3n}}{p+3} +$ &c. $= \frac{v^m}{p} - \frac{mv^{m-1}cx^n}{p.p+1} + \frac{m.m-1}{p.p+1}\times\frac{v^{m-2}c^2x^{2n}}{p+2} - \frac{m.m-1.m-2.v^{m-3}c^3x^{3n}}{p.p+1.p+2.p+3}$, &c. and the Fluent or Area above found $= \frac{dx^{pn}}{n}$ into $\frac{v^m}{p} - \frac{mv^{m-1}cx^n}{p.p+1} + \frac{m.m-1.v^{m-2}c^2x^{2n}}{p.p+1.p+2}$, &c. $= \frac{dx^{pn}v^m}{np}$ into $1 - \frac{m}{p+1}\times\frac{cx^n}{v} + \frac{m.m-1}{p+1.p+2}\times\frac{c^2x^{2n}}{v^2} - \frac{m.m-1.m-2}{p+1.p+2.p+3}\times\frac{c^3x^{3n}}{v^3}$, &c. Q. E. I.

Co-

COROLLARY 1.

Hence may the Fluent of $\overline{a+cx^n}|^{\frac{-m}{r}} \times dx^{\frac{pn}{r}-n-1}\dot{x}$ be easily derived; for let $\frac{p}{r}-n-1$ and $-\frac{m}{r}$ be substituted in Place of p and m respectively, and it will become $\frac{dv^{\frac{-m}{r}}x^{\frac{pn}{r}-n}}{\frac{pn}{r}-n}$ into $1+\frac{m}{p}\times\frac{cx^n}{v}+\frac{m}{p}\cdot\frac{m+r}{p+r}\times\frac{c^2x^{2n}}{v^2}+\frac{m}{p}\cdot\frac{m+r}{p+r}\cdot\frac{m+2r}{p+2r}\times\frac{c^3x^{3n}}{v^3}+$ &c. for the Value sought.

COROLLARY 2.

If $a=0$, and $c=1$; therefore $v=x^n$ and $x=v^{\frac{1}{n}}$; the Fluent of $dv^{\frac{p-m-2r}{r}}\dot{v}$, or the Sum of the Series $\frac{drv^{\frac{p-m-r}{r}}}{p-r}$ into $1+\frac{m}{p}+\frac{m}{p}\cdot\frac{m+r}{p+r}+\frac{m}{p}\cdot\frac{m+r}{p+r}\cdot\frac{m+2r}{p+2r}+$ &c. $=\frac{drv^{\frac{p-m-r}{r}}}{p-m-r}$: But when $v=1$; $d+\frac{dm}{p}+\frac{d.m.\overline{m+r}}{p.\overline{p+r}}+\frac{d.m.\overline{m+r}.\overline{m+2r}}{p.\overline{p+r}.\overline{p+2r}}+$ &c. $=\frac{d.\overline{p-r}}{p-m-r}$; and by Transposition of the Terms, it will appear, that $\frac{m}{p}+\frac{m}{p}\cdot\frac{m+r}{p+r}+\frac{m}{p}\cdot\frac{m+r}{p+r}\cdot\frac{m+2r}{p+2r}+$ &c. $=\frac{m}{p-m-r}$;

$\frac{m}{p-m-r}$; consequently $\frac{m+r}{p+r}+\frac{m+r}{p+r}\cdot\frac{m+2r}{p+2r}+\frac{m+r}{p+r}\cdot\frac{m+2r}{p+2r}\cdot\frac{m+3r}{p+3r}+$ &c. $=\overline{\frac{m}{p-m-r}-\frac{m}{p}}\times\frac{p}{m}$ $=\frac{p}{p-m-r}-1=\frac{m+r}{p-m-r}$; and when $m=0$, and $r=1$; $\frac{1}{p+1}+\frac{1.\ 2}{p+1.p+2}+\frac{1.\ 2.\ 3}{p+1.p+2.p+3}$, &c. $=\frac{1}{p-1}$.

PROBLEM XXXVI. *Answered by* Rosnillon.

It is evident that the given Series may be resolved into the following ones, that is,

$$mnp \text{ into } \frac{1}{r}+\frac{1}{r^2}+\frac{1}{r^3}+\frac{1}{r^4}+\frac{1}{r^5}+\frac{1}{r^6}+\text{ \&c.}$$

$$\overline{np+mp+mn} \text{ into } \frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\frac{6}{r^6}+\text{ \&c.}$$

$$+\overline{p+n+m} \text{ into } \frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}+\frac{36}{r^6}+\text{ \&c.}$$

$$-1 \text{ into } \frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}+\frac{64}{r^4}+\frac{125}{r^5}+\frac{216}{r^6}+\text{ \&c.}$$

whose Sum will be found, supposing them to be continued *ad infinitum*, to be equal to $\frac{mnp}{r-1}$

$$+\frac{\overline{mn+pn+pm}\times r}{\overline{r-1}|^2}+\frac{\overline{p+n+m}\times r}{\overline{r-1}|^2}+\frac{\overline{p+n+m}\times 2r}{\overline{r-1}|^3}-\frac{r}{\overline{r-1}|^2}-\frac{6r}{\overline{r-1}|^3}-\frac{6r}{\overline{r-1}|^4}=\frac{mnp}{r-1}+\frac{\overline{mn+pn+pm-p-n-m+1}\times r}{\overline{r-1}|^2}+\frac{\overline{p+n+m-3}\times 2r}{\overline{r-1}|^3}$$

$-\frac{6r}{\overline{r-1}|^4}$. But the above Series after the mth Term may be resolved into the following ones, that is,

$\frac{pm-pn+mn-m^2}{r^m}$ into

$$\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}+\frac{6}{r^6}+ \&c.\ ad\ inf.$$

$\frac{p+n-2m}{r^m}$ into

$$\frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}+\frac{36}{r^6}+ \&c.\ ad\ inf.$$

$-\frac{1}{r^m}$ into

$$\frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}+\frac{64}{r^4}+\frac{125}{r^5}+\frac{216}{r^6}+ \&c.\ ad\ inf.$$

whereof the Sum will be found to be

$$\frac{\overline{pm+mn+p+n-pn-m^2-2m-1}\times r^{1-m}}{\overline{r-1}|^2}+$$

$\frac{\overline{p+n-2m-3}\times 2r^{1-m}}{\overline{r-1}|^3}-\frac{6r^{1-m}}{\overline{r-1}|^4}$; which, taken from the Series before found, leaves $\frac{mnp}{r-1}-$

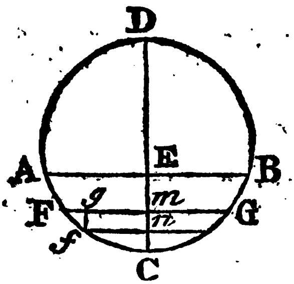
$$\frac{\overline{mnp-m-1}\times\overline{n-1}\times\overline{p-1}\times r+\overline{n-m-1}\times\overline{p-m-1}\times r^{1-m}}{\overline{r-1}|^2}$$

$$+\frac{\overline{p+n+m-3}\times 2r+\overline{2m-p-n+3}\times 2r^{1-m}}{\overline{r-1}|^3}+$$

$\frac{6r^{1-m}-6r}{\overline{r-1}|^4}=$ the required Value.

Problem XXXVII. *Answered by* John Turner.

Since CE=10, and BE=15, the Diameter CD will be 32.5, which being put $=a$, and CE$=b$, also any Distance C$m=x$, we shall have F$m=\sqrt{ax-xx}$,



and consequently the Area of the Surface of the Liquor at the Height C$m=p\times\overline{ax-xx}$ (p being $=3.1416$;) which multiplied by $\dot{x}$ gives $p\times\overline{ax\dot{x}-x^2\dot{x}}$ for the Fluxion of the Solidity; therefore $p\times\overline{ax^{\frac{1}{2}}\dot{x}-x^{\frac{3}{2}}\dot{x}}$ will be the Fluxion of the Time of Descent when the upper Cock is stopped, whose Fluent, when $x=b$, will become $p\times\overline{\frac{2ab^{\frac{3}{2}}}{3}-\frac{2b^{\frac{5}{2}}}{5}}$. But this Time is to the Time of Vacuation supposing the Velocity to continue the same as at the Beginning, as $20a-12b$ to $15a-10b$, or as 1 to .7311; therefore, if half the Content of the Segment ACB be denoted by n, we shall have $2.735n$ for the Quantity of Water run out of the lower Cock, with the first Velocity, in one Hour.

Now let both the Cocks be opened together, and let the Time of descending the Distance Em be denoted by T; then the Quantity of Liquor running in at the upper Cock, in the Time $\dot{T}$, will be $n\dot{T}$, and that running out at the lower Cock, in the same

Time, will be $\frac{2.735x^{\frac{1}{2}}n\dot{T}}{\sqrt{b}}$; therefore the Decreaſe of Liquor in the ſame Time will be denoted by $\frac{2.735x^{\frac{1}{2}}n\dot{T}}{\sqrt{b}} - n\dot{T}$, which is manifeſtly equal to $p \times ax\dot{x} - x^2\dot{x}$, and therefore $\dot{T} = \frac{pb^{\frac{1}{2}}}{nd} \times \frac{ax\dot{x} - x^2\dot{x}}{x^{\frac{1}{2}} - \frac{b^{\frac{1}{2}}}{d}}$;

whoſe Fluent, by putting $z = x^{\frac{1}{2}} - \frac{b^{\frac{1}{2}}}{d}$, will be found to be $\frac{pb^{\frac{1}{2}}}{nd}$ into $\frac{2az^3}{3} + \frac{3ab^{\frac{1}{2}}z^2}{d} + \frac{6abz}{d^2} + \frac{2ab^{\frac{3}{2}}}{d^3} \times L.z - \frac{2z^5}{5} - \frac{5b^{\frac{1}{2}}z^4}{2d} - \frac{20bz^3}{3d^2} - \frac{10b^{\frac{3}{2}}z^2}{d^3} - \frac{10b^2z}{d^4} - \frac{2b^{\frac{5}{2}}}{d^5} \times L.z$.

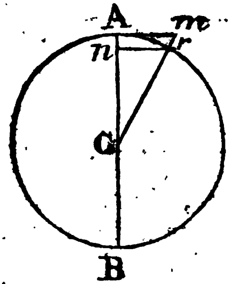
Problem XXXVIII. *Anſwered by* John Turner.

Since it is proved, (in *p.* 149, *Simpſon*'s *Fluxions*) that when the Gravity at the Equator is 1, the Centrifugal Force will be $\frac{1}{289}$, and that the Central Force under the Equator is to that in any given Latitude as the Square of Radius to the Square of the Coſine of the ſaid Latitude, we ſhall have, 0013398 equal to the Central Force in 51°. 30′, which being taken from the Attraction gives, 9986602 for the Force with which the Pendulum in the propoſed Latitude is actuated. But the Number of Vibrations performed by equal Pendulums in the ſame Time is in the ſubduplicate Ratio of the Forces, therefore $\sqrt{.9986602}$ (the ſquare Root of the Force when the Earth is in Motion) : $\sqrt{1}$ :: 24×3600 (=86400) : 86457.64 the Number of Seconds that would be performed in 24 Hours if the Earth's Rotation was to

to ceafe; therefore the Number of Vibrations or Seconds gained will be 86457.64—86400 (57.64) $=57''.38'''$.

Otherwise, by Samuel Farrer.

Let $ArBA$ reprefent the Earth confidered as Spherical: Put the Axis $AB(=2AC)$ about which the Earth is fuppofed to revolve, $=a$; the Diftance An which a heavy Body defcends, by means of its own Gravity, the firft Second $=d$; the Time of one Revolution in Seconds $=m$; the Periphery of a Circle whofe Diameter is Unity $=p$; the Cofine, to the Radius 1, of the propofed Latitude $=c$, and the abfolute Gravity $=G$: Then (by Cor. 9. Prop. 4. of *Newton*'s *Principia*) the Arch Ar which a Body



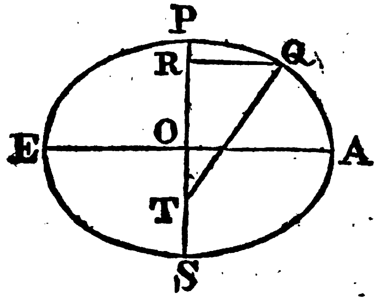
uniformly defcribes, in the Periphery of the Circle $ArBA$, made by a Plane paffing thro' the Center, in order that the centrifugal Force at the Equator, may be equal to the Force of Gravity, will be $\sqrt{da}$, and the Arch defcribed in the fame Time arifing from the Earth's prefent Rotation $=\frac{pa}{m}$; wherefore (by Cor. 1. of the before mentioned Prop.) $\frac{da}{AC}$: $\frac{p^2a^2}{m^2 \times AC}$ :: G : $\frac{p^2aG}{m^2d}$ the centrifugal Force at the Equator;

Equator; but this centrifugal Force, is to that in any other Latitude, in the duplicate Proportion, of the Radius, to the Cosine of the Latitude; therefore $\frac{c^2p^2aG}{m^2d}$ will be the centrifugal Force in the proposed Latitude; which being taken from G gives $G-\frac{c^2p^2aG}{m^2d}$ the accelerative Force of Gravity in that Latitude, when the Earth is in Motion; but the Number of Vibrations, made in the same Time by equal Pendulums, acted on by unequal Forces, being directly in the subduplicate Proportion of those Forces; it follows that $\sqrt{G-\frac{c^2p^2aG}{m^2d}} : \sqrt{G} :: m$ or 86400 the Number of Seconds, or Vibrations, in 24 Hours, when the Earth is in Motion, to 86457,9225 the Number of Vibrations which would be performed in the same Time, was the Earth's Rotation to cease intirely; therefore if from this last Number, 86400 be subtracted, the Remainder 57,9225, will, it is manifest, be the Number of Seconds the Pendulum would gain in one Day. *Q. E. F.*

Problem XXXIX. *Answered by* John Turner.

The Ratio of the Equatorial Diameter to the Polar being given as 2 to 1, if x be put for the latter, the former will be $2x$, and therefore the Content of the Spheroid as $4x^3$; which, if d be put for the Diameter of the Earth considered as a Sphere, will be $=d^3$, consequently $PS=\frac{d}{\sqrt[3]{4}}$, and $EA=\frac{2d}{\sqrt[3]{4}}$; which, when d is taken $=2$, will come out equal to 1,2599 and 2,5198 respectively.

Let

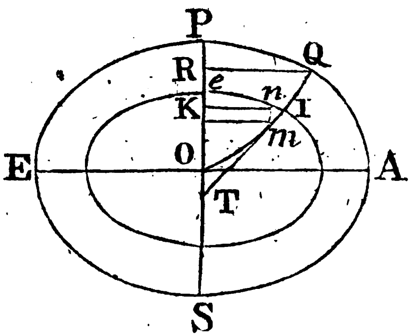


Let now $OP = .6299 = a$, $OR = z$, and the Tangent of the Angle RTQ the Complement of the given Latitude $= t$; then by the Property of the Ellipsis we have $RQ = 2\sqrt{a^2 - z^2}$, and $RT = 4z$; therefore, since $RT : RQ ::$ Rad. : Tangent QTR, we have $4tz = 2\sqrt{a^2 - z^2}$, and consequently $z = \frac{a}{\sqrt{4tt+1}} = .24372$; whence QR and QT are known.

But, *by Simpson's Dissertations*, it is proved that the Gravitation when the Earth is at rest, under the Form of a Sphere, will be to the Gravitation in the proposed Latitude, when in a Spheroid, as 1 to $QT \times .3954$.

And since, *by the Question*, the Body is supposed to descend (along a Curve or an inclined Plane) every where in the Direction of its own Gravity, it is evident that the Path QIO, of the Body must be such, as to be perpendicular to every Ellipsis similar and concentric to EPAS; which Path, therefore, will be found to be a Parabola, whose Equation is $y^m = z \times \frac{c^m}{b}$, where $RO = b$, $RQ = c$, any Abscissa $OK = z$, and its corresponding Ordinate $KI = y$, m being to Unity as the Square of the greater Diameter to that of

of the leſſer; which Curve, therefore, in the Caſe



propoſed will be defined by $y^4 = z \times \frac{c^4}{b}$, or $z = \frac{by^4}{c^4}$, whence $KT = 4z = \frac{4by^4}{c^4}$, $IT = \frac{y}{c^4}\sqrt{c^8 + 16b^2y^6}$; therefore by ſimilar Triangles $KI : In :: IT : Im = \frac{\dot{y}}{c^4}\sqrt{c^8 + 16b^2y^6}$.

Now, IT being ſuppoſed to repreſent the Force in the Direction of the Curve IO, we ſhall have $\frac{y}{c^4} \times \sqrt{c^8 + 16b^2y^6} \times \frac{\dot{y}}{c^4}\sqrt{c^8 + 16b^2y^6} = -v\dot{v}$, v being ſuppoſed equal to the Velocity in the Point I, therefore $\frac{y^2}{2} + \frac{2b^2y^8}{c^8} = -\frac{v^2}{2} + d$; but when $y = c$, $v = 0$, therefore $\frac{c^2}{2} + 2b^2 = d$; conſequently $v^2 = c^2 + 4b^2 - y^2 - \frac{4b^2y^8}{c^8}$, and $v = \sqrt{c^2 + 4b^2 - y^2 - \frac{4b^2y^8}{c^8}}$,

whence $\frac{\dot{y}\sqrt{c^8+16b^2y^6}}{c^4\sqrt{c^2+4b^2-y^2-\frac{4b^2y^8}{c^8}}}$ will be the Fluxion of the Time required.

PROBLEM XL. *Answered by* John Turner.

It appears, *by Proposition 2d, Page 3d, Simpson's Essays*, that the Part which a Star through the Aberration appears to describe, in a Plane parallel to the Ecliptic, will be a Circle, let the Eccentricity of the Earth's Orbit be what it will. Therefore, if the Ellipsis in which the Earth moves be supposed to degenerate to a Parabola or an Hyperbola, the Path will still, it is manifest, be either a whole Circle, or a Portion of a Circle; that is, a Circle in the Parabola where the true Place of the Star will be in the Periphery, and a Portion thereof in the Hyperbola, where the true Place of the Star will be without the Periphery. Therefore, since the Projection of a Circle upon any oblique Plain is either a Circle, Ellipsis, or a Right one, it follows that, the required Path will be always one of these three; that is, a Circle when the Star is in the Pole of the Ecliptic, a right Line when in the Ecliptic itself, and an Ellipsis in all other Cases.

A
COLLECTION
OF
PROBLEMS
TO BE
Answered in the next NUMBER.

PROBLEM XLI. by *Spicelogos*.

TWO Notes, valued together at 308 *l.* 6 *s.* 8 *d.* one whereof was due at the End of 6 Months, the other at 8, at a certain Rate *per Cent.* were presented to a Banker, to be discounted, who received 8 *l.* 6 *s.* 8 *d.* as a Premium; the Interest of the two Sums, each for its respective Time, will amount to 4 *s.* 8 *d.* $\frac{3}{4}$ more than the Discount; from whence the Value of each Note and the Rate of Interest may be found, and are here required.

PRO-

PROBLEM XLII. by *Samuel Hill.*

Our Ship coming into an unknown Depth of Water, we threw out an Anchor; when that was at the Ground we were close to a Buoy; then, veering away 50 Fathom of Cable more, made us 70 Fathom from the Buoy; Quere, the Depth of Water?

PROBLEM XLIII. by *Samuel Farrer.*

A Ship sails from a certain Port, upon a S. W. Course, at the rate of 4 Miles *per* Hour; after she has sailed 5 Hours, another Ship, from a Port 22 Miles North of the former, sets out in Pursuit of her, sailing at the rate of 5 Miles *per* Hour: Now it is required to find what Course the second Ship must steer to come up with the first in the shortest Time possible?

PROBLEM XLIV. by *John Moor.*

The Perimeter, and all the Angles, of any plain Triangle, being given; to determine the Sides?

PROBLEM XLV. by *Nath. Season.*

To draw a Right-Line, to make a given Angle with one of the Sides of a given Triangle, so that the Triangle, cut off thereby, may be to the whole in a given Ratio.

PROBLEM XLVI. by *Samuel Farrer.*

In a given Pentagon, to inscribe a Square.

PROBLEM XLVII. by *Samuel Farrer.*

In a given Portion of a Circle, to inscribe a Rectangle, whose Length shall be to its Breadth, in a given Proportion.

PRO-

PROBLEM XLVIII. by *Samuel Farrer*.

Three Right-Lines, drawn from the angular Points, terminating in, and bisecting the opposite Sides of any plain Triangle, being given; to construct the Triangle.

PROBLEM XLIX. by *John Turner*.

The Perimeter, Area, and one of the Angles, of any plain Triangle, being given; to construct the Triangle.

PROBLEM L. by *Samuel Farrer*.

If from a Point, either within, or without, any right-lined Figure whatsoever, Perpendiculars be let fall on every Side; then the Squares of the Segments, made by those Perpendiculars, alternately taken, are equal. *Quære*, the Demonstration?

PROBLEM LI. by *Ticho Oxoniensis*.

If two right-angled spherical Triangles, have one acute Angle common to both, and the Difference of their Bases be bisected by a Perpendicular; it will be, as the Tangent of the lesser Perpendicular, is to that of the greater :: Rad. : Tangent of an Arc; and as Rad. : to the Tangent of the Excess of this Arch above $45°$ :: Tangent of the Distance of the Point of Bisection from the common acute Angle : to the Tangent of $\frac{1}{2}$ the Difference of their Bases.

PROBLEM LII. by *Ticho Oxoniensis*.

If two Lines drawn from the Curve, to the Focus of a Parabola, be given both in Length and Position; the Position of the Axis may be determined by

by the following Proportion, *viz.* as the ſquare Root of the leſſer Line : to that of the greater :: Rad : the Tangent of an Arc ; and as Radius : the Tangent of the Exceſs of the ſaid Arc above 45° :: the Cotangent of the fourth Part of the Angle, included by the given Lines : the Tangent of a ſecond Arch ; then if the half of the Angle included by the given Lines, be taken from, or added to, the double of the laſt found Arch ; the Remainder, or Sum, will give the Angle formed by the Axis, with the leſſer, or greater, Line reſpectively.

PROBLEM LIII. by *Samuel Farrer.*

What is the Odds of not throwing, either 35, 36, 37, or 38, once in three Trials, with ten Dice.

PROBLEM LIV. by *John Turner.*

If in a right-angled Triangle, the Lengths of two Right-Lines, drawn from the Extremities of the Hypothenuſe, terminating in the oppoſite Sides at equal Diſtances from the right Angle, and interſecting in a Perpendicular, falling from the right Angle upon the Hypothenuſe, be given ; 'tis required to determine the Triangle.

PROBLEM LV. by *John Turner.*

The Diſtance and Poſition of two Boats, whereof one is to croſs a River, and the other to ſail directly down it, being given ; 'tis required to find in what Direction the firſt muſt croſs the River, ſo that it may be as much before the other as poſſible, ſuppoſing the Velocity of the firſt to be to that of the laſt in a given Ratio ?

PROBLEM LVI. by *Samuel Farrer*.

Of all the equal headed Casks, that can possibly be cut out of a given Spheroid; to find that, which being filled with Liquor and placed with its Axis perpendicular to the Horizon, will require more Time, to empty itself at a given Hole in its Base, than any other?

PROBLEM LVII. by *John Turner*.

To find the Equation of that Curve, which shall cut an infinite Number of Ellipses, similar and concentric to a given one; so as to be perpendicular to them all.

PROBLEM LVIII.

If two Bodies L and T, whose Masses are respectively equal to those of the Moon and Earth, were projected at the same Time, and in the same Plane, from two Places A and B, at the Distance of 100000 Miles from each other; the former, L, with a Velocity of 5 Miles *per* Second, making an Angle with AB of 100 Degrees, and the latter with a Velocity of 2 Miles *per* Second, making an Angle (on the same Side AB) of 60 Degrees; 'tis required to find the Distance and Position of the two Bodies with respect to the Points A and B, also with respect to each other, after they have been 48 Hours in Motion, supposing them, when in Motion, to be only acted upon by each other.

The End of NUMBER III.

THE

Mathematician.

DISSERTATION IV.

Upon the Progreſs and Improvements of GEOMETRY.

Containing,

A familiar and perſpicuous Account of the Nature of Fluxions, and the Method of aſſigning their Relations by finite Magnitudes.

HAVING in our laſt given ſome Account of Dr. *Barrow*'s Improvements in Geometry, by conceiving the Generation of Magnitudes by local Motion, from whence, in one Inſtance, we ſhewed how he determined the Ratio of the Velocities of thoſe Motions; we do not think it proper now, to retard the Expectation of our Readers, by calling back their Attention to the ſeveral minute Improvements introduced into the Geometry of curve Lines by

Des Cartes, *Fermat*, *Huddenius*, *Lucas Vallerius*, *Gregory of St. Vincent*, &c. in conſequence of the Method of Indiviſibles, invented by *Cavallerius*, and which in our firſt Diſſertation, was ſhewn to be inaccurate and ungeometrical: But having proceeded thus far, and being now arrived at the very Borders of the Doctrine of Fluxions, the moſt noble and ſublime Improvement that ever Geometry attained, a Method of moſt univerſal and comprehenſive Reaſoning, in a new and ſubtle Manner, by which the moſt remote and uſeful Properties and Relations of Quantities are diſcovered, and which contains all the omitted Improvements of the above mentioned Geometers, and infinitely more; we ſhall therefore in the enſuing Diſcourſe attempt an arduous Taſk, and endeavour to the utmoſt of our Abilities to ſhew, that in the Doctrine of Fluxions the great Author walked not upon inchanted Ground; but as ſolid geometrical *Terra firma* as ever *Euclid* and *Archimedes* trod; whoſe Methods of Demonſtration have ſtood the Teſt of 2000 Years; and though this laſt Method has of late been formidably attacked, and charged with Abſurdity and Inconſiſtency, yet has it been defended by ſome able Champions, and proved to be ſtrictly geometrical and ſcientific. The End we propoſe at preſent, is, 1ſt, To give an Idea and Definition of Fluxions. 2dly, To ſhew the Manner of determining what thoſe finite Magnitudes are, by which their Relations or Ratios may be expreſſed: And 3dly, To explain by what Methods and to what uſeful Purpoſes they may be applied. In proſecuting ſuch a Deſign, we flatter ourſelves, that every candid Geometrician will allow ſuch Indulgence to our Defects and Imperfections, as human Frailty, and the Nature of ſuch abſtruſe and metaphyſical Subjects require.

To avoid ſome Inaccuracies attending the Doctrine of former Geometers concerning curve Lines and

curvilinear Spaces, Sir *Isaac Newton* laid down or presuppofed this Poſtulate, *viz.* That Magnitudes are to be conſidered, not under the Notion of being increaſed, by a repeated Acceſſion of Parts, but as generated by a continued local Motion, or Flux of their elemental Parts: In Geometry, all Degrees of Magnitude may be produced, and in ſuch a way as may found a general Method of deriving their Affections from their Geneſis, we conceive the Quantities to be increaſed and diminiſhed, or to be wholly generated by Motion, or by a continual Flux analogous to it. The Quantities thus generated are ſaid to flow, and called *Fluents*, or *flowing Quantities*, and ſometimes *variable* or *indeterminate Quantities*, becauſe they are capable of receiving an indefinite Number of particular Values, in a regular Order of Succeſſion. He next diſcovered a Method of determining and comparing the Velocities wherewith homogeneous Magnitudes increaſe, and called it the *Method of Fluxions*, from the equable Flux of Time, which we conceive to be generated by the continual Acceſſion of new Particles or Moments.

In this Method, geometrical Quantities are not preſented to the Mind as compleatly formed at once; but as riſing gradually before the Imagination, by the Motion of ſome of their Extremes: For local Motion ſuppoſes a Notion of Time, and Time implies a Succeſſion of Ideas. We eaſily diſtinguiſh into what was, what is, and what will be, in theſe Generations of Quantities; and ſo we commodiouſly conſider thoſe Things by Parts, which would be too much for our Faculties, and extremely difficult for the Mind to take in the whole together, without ſuch artificial Partitions and Diſtributions.

Sir *Isaac* makes this Suppoſition, which may eaſily be granted, *viz.* That a Line may be conceived to be traced out gradually by a Point, moving either with

an equable Motion, or with a Velocity accelerated or retarded; and the Velocity or Degree of Swiftness which this Point moves with, in any Part of the Line described, or at any Moment of the Time of Description, is the Fluxion of the Line at that Place, or at that Moment of Time. In like Manner Surfaces are generated by the Motion of Lines; Solids, by the Motion of Surfaces; Angles, by the Rotation of their Sides. These Generations of Quantities we daily see to obtain in *rerum naturâ*, and is the Manner the antient Geometricians had often Recourse to, in considering their Production, and then deducing their Properties from such actual Descriptions. But the Velocity with which a Surface flows, or its Fluxion, is not literally and strictly the Velocity of the generating Line, or of any of the particular Parts or Terms of the Surface, but the Celerity or Degree of Swiftness wherewith its whole Magnitude is changed; being the same as the Velocity of a given right Line, which by moving parallel to itself, is supposed to generate a Rectangle, which is always equal to the Surface; for, in Reality, it is the Rate of its Increase, or that Velocity, however expressed, with which the Space at all Times constantly increases, and denotes the Degree wherewith this Augmentation in every Moment proceeds. So the Velocity with which a Solid flows, or its Fluxion, is the same as the Velocity of a given plain Surface; that by moving parallel to itself, is supposed to generate an erect Prism or Cylinder, that is always equal to the Solid. The Velocity with which an Angle flows, or its Fluxion, is the same as the Velocity of a Point that is supposed to describe the Arch of a given Circle, which always subtends the Angle and measures it.

Hence

Hence this general Definition.

Fluxions are not Magnitudes, but the Velocities with which Magnitudes varying by a conſtant Motion increaſe or diminiſh; and the Magnitudes themſelves are called the Fluents of thoſe Fluxions.

And that this Account is conſiſtent with Sir *Iſaac*'s Meaning, we ſubmit to any Reader who will conſider the Author's own Words, in his Introduction to his Treatiſe of Quadratures, where he ſays, *Quantitates mathematicas, non ut ex partibus quam minimis conſtantes, ſed ut motu continuo deſcriptas hic conſidero. Lineæ deſcribuntur, ac deſcribendo generantur; non per appoſitionem partium, ſed per motum continuum punctorum, ſuperficies per motum continuum linearum, &c. Conſiderando igitur quod quantitates æqualibus temporibus creſcentes, & creſcendo genitæ, pro velocitate majori vel minori qua creſcunt ac generantur, evadunt majores vel minores; methodum quærebam determinandi quantitates ex velocitatibus motuum vel incrementorum, quibus generantur, & has motuum vel incrementorum velocitates nominando fluxiones, & quantitates genitas nominando fluentes, &c.* viz. I conſider mathematical Quantities in this Place, not as conſiſting of very ſmall Parts; but as deſcribed by a continued Motion. Lines are deſcribed, and thereby generated, not by the Appoſition of Parts, but by the continued Motion of Points; Superficies by the Motion of Lines, *&c.* Therefore conſidering that Quantities which increaſe in equal Times, and by increaſing are generated; I ſought a Method of defining Quantities from the Velocities of the Motions or Increments with which they are generated, and calling theſe Velocities of the Motions or Increments *Fluxions*, and the generated Quantities *Fluents*, &c.

Hence it appears they have a falſe Idea of this Doctrine, who ſuppoſe a Fluxion to be a *compleat*

Part

Part of a flowing-Quantity, and that an *Infinity of Fluxions* conſtitutes it.

Tho' theſe Velocities, abſolutely conſidered, are abſtract Ideas only, or Modes of Motion, yet are they mathematical Quantities, being ſuſceptible of indefinite Gradations, may be intended or remitted, increaſed or diminiſhed in different Parts of the Space deſcribed, according to an indefinite Variety of ſtated Laws: And it is plain, that the Space thus deſcribed, and the Law of Acceleration or Retardation (*i. e.* the Velocity at every Point of Time) muſt have a mutual Relation to each other, and muſt mutually determine each other; ſo that one of them being aſſigned, the other by neceſſary Inference may be derived from it. In order therefore to repreſent the Quantity of theſe Velocities to the Mind, we muſt make Uſe of ſome regular Magnitudes proportional to them in their Stead, the Difference and Proportion of whoſe Parts we can eaſily diſcover: And as a right Line is moſt ſimple and perſpicuous of any, 'tis therefore the fitteſt to repreſent any Degree thereof. This may be done commodiouſly enough, by the Aſſiſtance of that well known Principle in Mechanics, *viz.* That when the Times are equal, the uniform Velocities are as the Spaces deſcribed. So now, ſince Spaces may be well denoted by Lines, we have known Magnitudes whereby we can expreſs the Ratio of theſe Velocities or Fluxions, and to compute the Quantity of thoſe Ratios from.

We are further to obſerve, that tho' the Fluxion of a Space, from what has been above ſaid, is not ſo ſimple and eaſy to be conceived as that of a Line, yet it is poſſible to reduce the Fluxions of all other flowing Quantities to this of a Line, *viz.* by imagining a Point ſo to paſs over any ſtreight Line, that its Length meaſured out, while the other flowing Quantity is deſcribing, ſhall augment in ſuch Proportion

Proportion as that flowing Quantity; so that the Fluxion, or Velocity, or Rate of Increase of this Fluent, will ever be proportional to the actual Velocity of the Point describing the Line: Neither is any other Thing necessary, than to determine the Velocity wherewith such Lines as these are described, in all Application of Fluxions to geometrical Problems, where Spaces are concerned.

This brings us to the second Part of our Design, namely, *To shew the Manner of determining those finite Magnitudes, by which the Relations of Fluxions may be expressed.* And here,

The great Inventor lays down the following Proposition as a fundamental one, *viz.* "Fluxions "are very nearly, as the Augments of their Fluents "generated in equal, but very small Particles of "Time; and to speak accurately, they are in the "first Ratio of the nascent Augments, or the last "Ratio of the evanescent Parts: But they may be "expounded by any Lines which are proportional "to them."

Before we enter on a Discussion of this Proposition, it will be proper to observe, 1st. That the Fluxions of flowing Quantities, being the Celerities with which they are supposed to flow, and by which they are generated; the Fluxion of any variable Quantity ought never to be considered *absolutely*, or by itself, but *relatively*, or with Relation to the Fluxion of some other flowing Quantity of the same kind. So that if at any Time, for Brevity sake, the Fluxion of a Quantity be mentioned absolutely; yet there is always a *supposed* Relation to the Fluxion of some other Quantity, with which it is understood to be compared; and ordinarily in the Comparison, one of the flowing Quantities is supposed to flow equably and uniformly; so that its Fluxion being constant and invariable, is considered as a Standard or Measure, to which the Fluxions of the other variable Quantities are referred.

2dly,

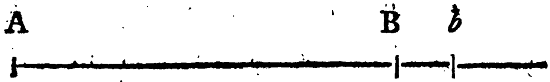
2dly. That if the Fluxion of any flowing Quantity, or its Celerity of flowing by continually varying, *i. e.* continually accelerated or retarded for a Time, the Fluxion in any one Place, or at any one Inſtant of that Time, is different from the Fluxion of that Quantity in any other Place, or at any other Inſtant of Time. For whatever is continually varying, by Increaſe or Decreaſe, by the very Suppoſition, muſt be of a different Value in every different Place, or at every different Inſtant of Time; otherwiſe it would not be continually varying. To illuſtrate this in the Caſe of a falling Body: The Velocity with which it falls in any one Place, or at any one Inſtant of Time during its Fall, is different from the Velocity it hath in any other Place, or at any other Inſtant of Time; and it is all one whether the Motion be uniformly accelerated or retarded; or whether it be done according to any other Law than what obtains at preſent.

In order to comprehend more clearly the Truth of the Propoſition above, it will be neceſſary to premiſe a few Lemmas reſpecting Motion.

1. If two Bodies moving uniformly deſcribe Spaces denoted by S and $ſ$ in the Times T, and t with the Velocities V and v (where the Capitals ſignify the greater Space, Time and Velocity, and the ſmall Letters the leſſer) then from the Nature of Motion, as may be ſeen in our ſecond Diſſertation $S=TV$ and $S=tv$: *i. e.* ſuch Spaces may be juſtly expounded by a Rectangle of the Time and Velocity: Therefore $S : ſ :: TV : tv$, and if the Times of Deſcription are equal, *viz.* if $V=v$, then $T : t :: S : ſ$ & *è contra*; if the Velocities are equal, *viz.* if $V=v$, then $T : t :: S : ſ$ & *è contra*. If both Times and Velocities are equal, *viz.* if $V=v$ and $T=t$ then then $S=ſ$.

2. If the right Line AC be deſcribed by the Motion of the Point=B with a Velocity continually accelerated,

accelerated, the Spaces described in equal Times will still increase, as the Time from the Beginning of its Motion increases.



For if the Motion was equable or uniform, the Spaces described in equal Times would be equal, *per Lemma* 1st, therefore when it is continually accelerated, the Spaces described in equal Times must still encrease, the further the Time is from the Beginning of Motion.

3. If the Space or Augment B*b* be described by the Point B, moving with a Velocity continually accelerated or continually retarded in any Time, an uniform Velocity capable of describing the Space B*b* in the same Time, will fall in betwixt the two extreme Velocities, wherewith the Point B moves at the Beginning and End of the Space or Augment B*b*.

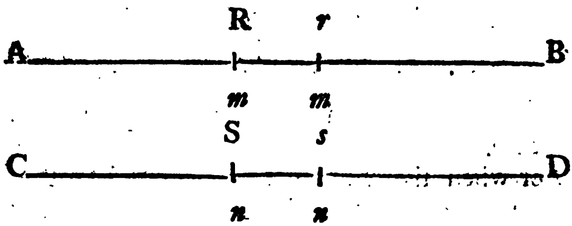
For when the Motion is accelebrated, if the Velocity with which the Point B moves at the Beginning of the Space B*b*, continued the same, it would not be capable of describing so great a Space as B*b*, in the same Time it is described by the accelerated Motion; but the Velocity with which B moves at the End of the Space B*b*, would produce a greater Space than B*b* in that Time. Therefore an equable or uniform Velocity capable of describing B*b* in the same Time it is described by an accelerated Velocity, must fall in betwixt the two, *i. e.* it must be greater than the former and less than the latter; since, when uniform Velocities or equable Motions are compared, and the Time the same, the Velocities are as the Space described directly; and it is evident, the like way of reasoning may be applied to Mo-

tions continually retarded, whence the Truth of the Lemma appears.

And now we apprehend the Way is sufficiently paved for the Proof of the first Part of the Proposition above concerning the Relation of Fluxions: And as this seems most clearly explained by Mr. *Simpson* in the first three Problems of his Treatise of Fluxions, first Edition, we'll take the Liberty to offer them to our Readers in this Place.

The Fluxions of variable Quantities, or their Relations to each other, are always measured and expressed by the finite Spaces that would be uniformly described in equal Times, with the Velocities by which those Quantities are generated.

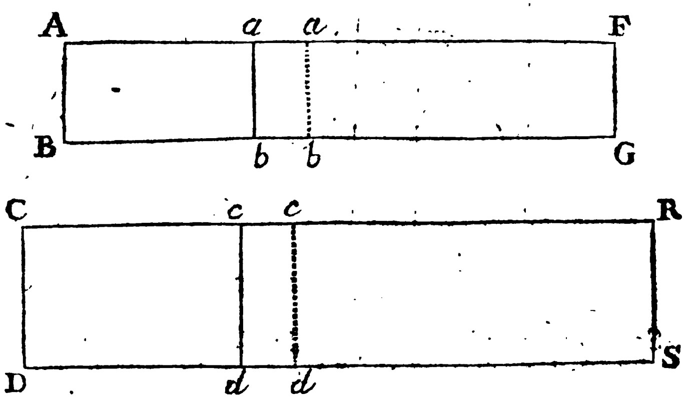
In order to determine what these finite Spaces are, and to assign their Proportion, let us imagine two Points *m*, *n*, to move at the same Time



from the Points A, C, towards the Points B, D: Then will any two Lines that are to each other as the Velocities with which those Points are carried in any two cotemporary Positions as R, S, be the Fluxions of the generated Lines, AR, CS. If the generating Points flow uniformly, any two Augments, Increments or Spaces as R*r*, S*s*, described in the same Time, will exactly express the Velocities of those Points, or the Fluxions of the Lines AR, CS. For by the Nature of uniform Motion and the 1st Lemma above, when $T = t$ then $V : v :: S : s$. Therefore the Ratio of R*r* to S*s* is the Ratio of the Fluxions required to be assigned.

In

In like Manner the Fluxion of two plane Surfaces or Superficies may be conceived by supposing two Lines, as *ab*, *cd* to move continually parallel to themselves, over the parallel and immoveable Lines AF, BG, and CR, DS. If the Spaces or

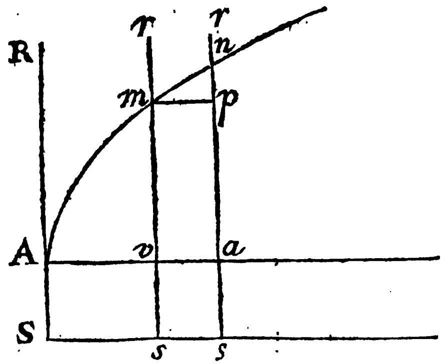


Augments or Increments, the Parallelograms *ab*, *cd*, be both described uniformly in equal Times by the Motion of the said generating Lines, they will accurately express the Fluxions of the generated Parallelograms A*b*, C*d*. And because *bb*, *dd*, the Fluxions of the Lines B*b*, D*d*, are as the Velocities with which the generating Lines are carried; therefore the Fluxion of any Space generated by the Motion of a Line always parallel to itself, will ever be exactly as the Length of the generating Line multiplied by its Velocity.

This also holds true in the Generation of Curves, where the Lengths of the generating Lines continually vary: *E. g.* Suppose the curve-lined Space A*mv* and the Parallleogram A*s* to be generated by the continued uniform Motion of the Line *rs* in the same Time; then will the Fluxions of these two Spaces be to each other, as the Parallelograms *ma* : *sa*, (being the simultaneous Increments or Augments)

 or

or as the generating Lines *mv* : *vs*; and because *vs* is invariable, the Fluxion of the curve-lined Space *Amv* will be always as the ordinate *vm*, because the



Space that would be uniformly described with the equable Velocity of the Line *vm*, by which the curve-lined Area is generated, cannot be proportional to any other Quantity, when the Abscissa *Av* flows uniformly.

Hitherto we have spoken concerning uniform Motions only, when both the generating Quantities flow equably; we shall in the next place shew how to assign the Fluxions from the known Relation of the Fluents, altho' one of them be any how accelerated or retarded; in that Case the Space or Increment S*s* will be either greater or less than that which would have been uniformly described in the same Time with the Velocity of the generating Point S: and therefore the Ratio of S*s* to R*r*, is greater or lesser than that of the Fluxion of CS to the Fluxion of AR. Suppose two Points *m* and *n* to move at the same Moment of Time from the Points A and C continually towards B and D, and the Velocity of the Point *n* to be such, that the Space described thereby, shall always be equal to some Power of the Space described by the other Point *m* moving uniformly, that is

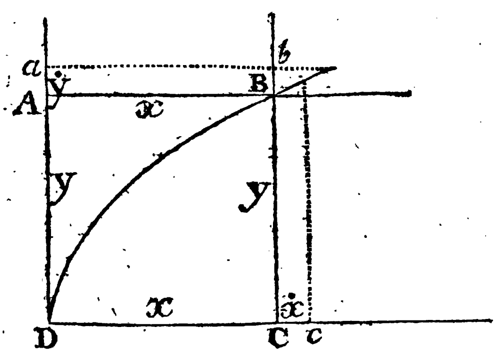
is if AR$=x$, then CS will be$=cx^v$ (where c represents some given Quantity and v any Number whatever.) Take any two other cotemporary Positions of these Points, as r, s: and put the Lines g and h, for the

Spaces that would be uniformly described together in the same Time, with the required Velocities of the Points in R and S; then these Lines which are as the Velocities themselves by Lemma 1st, will exactly express the required Fluxions of the Lines AR and CS.

Now to determine the Ratio of these Lines g to h, or of the Fluxions required, let us try if we can find two other finite Quantities in the same Ratio: Call R$r=p$. Then $c\times\overline{x-p}^{v}=Cs$. But by infinite Series and Sir *Isaac*'s Theorem $\overline{x-p}^{v}=x^{v}-vx^{v-1}p+v\times\frac{v-1}{2}x^{v-2}p^{2}$, &c. Therefore CS—C$s$=S$s$=$cvx^{v-1}p-v\times\frac{1-v}{2}cx^{v-2}pp$, &c. Because this Distance Ss is descricbed with a Velocity either continually accelerated or retarded, according as v is greater or less than Unity, it will consequently be equal to the Space that would be uniformly described in the same Time, with the Velocity that n hath at a Point, suppose e, posited somewhere near the Middle between s and S, and therefore *per* Lemma 1. as $p : cvx^{v-1}p-v\times\frac{v-1}{2}cx^{v-2}pp$ &c. (=Ss) :: g : $gcvx^{v-1}-v\times\frac{v-1}{2}pgcx^{v-2}$, &c.=the Space that would be uniformly described with the Velocity of the Point

Point S, in the ſame Time that m is moving over the Space g: Therefore when s, e, and S coincide, *viz.* when the Point n arrives at the immoveable Point S, we ſhall have $gcvx^{v-1} - v \times \frac{v-1}{2} pgcx^{v-2}$, &c. $= b$; but becauſe p ($=$Rr) then becomes $= o$, all the Terms multiplied thereby will entirely vaniſh; therefore $gcvx^{v-1} = b$; hence we have as $1 : cvx^{v-1} :: g : b$ or as $x : cvx^{v} :: g : b$, ſo that if $\dot{x}$ be made to repreſent g, then b or the required Fluxion of cx^{v} will be expounded by $cvx^{v-1}\dot{x}$, becauſe $x : \dot{x} :: cvx^{v} : cux^{v-1}\dot{x}$.

Having before proved that the Fluxion of a curve-lined Space is as a Parallelogram, whoſe Length is the Ordinate, and Breadth any Increment or Fluxion of the Abſciſs; let us now deduce the Fluxion of the Rectangle AC therefrom.



The Fluxion of the curve-lined Space DBC is as BC×C$c$$=y\dot{x}$, and the Fluxion of the curvelined Space DAB is as AB×A$a$$=x\dot{y}$.

But the Space DBC+DAB=Parallelogram AC; therefore $y\dot{x}+x\dot{y}$ is as the Fluxion of the Rectangle or Parallelogram AC.

And now we hope the firſt Part of Sir *Iſaac*'s Propoſition above, for finding the Relations of Flux-

ions

ions by Means of the Increments is sufficiently clear; hence the Fluxions of all other Quantities, however expressed, may be determined, as being deducible either from a Power or Rectangle, as may be found in all the practical Treatises on this Subject.

'Tis true, the last Example but one for assigning the Ratio of the Fluxions of Powers, is drawn not from the Consideration of Increments only, but is also founded on the Doctrine of prime and ultimate Ratios, hereafter explained; however, it is intelligible enough in this Place.

From such Considerations as these, in regard to the Generation of Magnitudes, Sir *Isaac* instituted an Analysis for this his very extensive and compendious Method of Calculation, which he readily applied in the Manner hereafter to be described, to the finding the *Maxima* and *Minima*, drawing Tangents, determining the Curvature of Curves, squaring curvilinear Surfaces, and to other Problems in the higher Geometry; all this he was Master of about the Year 1665, about which Time he introduced this Doctrine of Fluxions into his Calculations, that he might proceed without Indivisibles, as much as possible; because, as he says, we have no precise Ideas of infinitely little Quantities; nor can we pursue the variable and fleeting Forms of the inscribing and circumscribing Figures in *infinitum*; so that, when they should become equal to the Curve, they may not totally withdraw themselves from the Imagination, and all Idea about them be lost. But in determining the Proportions of these Fluxions, until about this Time, he still allowed himself some Use of infinitely little Quantities. No doubt, but upon reading the Ancients, he from thence would have been enabled to have demonstrated the Proportions of Fluxions according to their accurate Methods; for he did muchmore, in finding out one of his own,

own, *viz.* his Method of prime and ultimate Ratios, which is more compendious than theirs, and equally geometrical. This ſerved not only to demonſtrate the Proportions of Fluxions, but was applicable to the ſynthetic Demonſtration of all Propoſitions relating to Curves. When he diſcovered this Method we do not certainly know; but we are ſure he had Part of it in the Year 1669, on Account of a Demonſtration added to the End of his *Analyſis per Æquationes*, &c. which was ſent at that Time by Dr. *Barrow* to Mr. *Collins*. But moſt probably he had not then compleated this Method, ſince in the Lectures he read the ſame Year at *Cambridge* on his admirable Diſcoveries in Optics, he did uſe Indiviſibles in his Demonſtrations.

It was in 1686 he firſt diſcloſed his Doctrine of prime and ultimate Ratios, in his immortal Work of the *Principia naturalis Philoſophiæ mathematica.* It is ſurpizing with what Modeſty, as if it were with Fearfulneſs to offend ſuch as had been Admirers of Indiviſibles, he introduced ſo excellent and truly geometrical a Method, by cenſuring the other in the ſofteſt Manner. Tho' in anſwering the Objections that might be ſtarted againſt his own Method, he evidently proves, that he was fully apprized of the real Imperfections of Indiviſibles, at the ſame time ſhewing a Way to avoid them; yet he ſcarce condemns them himſelf, and frequently makes Uſe of Expreſſions peculiar to them, thinking it ſufficient once for all to inform thoſe who did not approve of Indiviſibles, how to correct ſuch Expreſſions, and render them conformable to his Method of prime and ultimate Ratios.

In the Year 1704, he publiſhed his Book of Quadratures, a Work worthy his profound Genius. He had now ſufficiently ſeen the Abuſes that had been made of infinitely ſmall Quantities, in what was called the differential Calculus; (of which you may

may find ſome Account in our firſt Diſſertation.) In the Introduction to this Book, he delivers a very diſtinct Account of his Method of Fluxions, and teaches how to find out their Proportions, by this Method of prime and ultimate Ratios; in order, as he ſays, to ſhew there was no Occaſion in the Uſe of Fluxions, to introduce infinitely little Quantities into Geometry; but ſtill ſaying, with his uſual Modeſty, that Errors might be avoided in the other Method, if we proceed cautiouſly. In all this plain Narrative of Matter of Fact, there appears no Inconſiſtency in Sir *Iſaac Newton*'s Account of his Methods, or the leaſt Shadow of his having been ever puzzled or confounded in his Ideas about them; as is miſrepreſented by the Author of the Analyſt, for want of his rightly underſtanding the Nature of theſe Inventions.

In order now to evince the Truth of the latter Part of the above Propoſition, for inveſtigating the Proportion of Fluxions accurately, it ſeems neceſſary that we ſhould explain what is meant by the Doctrine of prime and ultimate Ratios.

Sir *Iſaac Newton*'s 1ſt Lemma in the 1ſt Section of his Principia contains the Foundation of this Method, and runs in theſe Words; *Quantitates, ut & quantitatum rationes, quæ ad equalitatem tempore quovis finito conſtanter tendunt, & ante finem temporis illius propiùs ad invicem accedunt, quam pro datâ quavis differentiâ, fiunt ultimo æquales.* i. e. Quantities, as likewiſe the Ratios of Quantities, which conſtantly tend to Equality during any finite Time, and before the End of that Time come nearer to one another than by any given Difference, at laſt become equal.

If you deny it, ſays he, let them be at laſt unequal, and let their laſt Difference be D. Therefore they cannot come nearer to Equality than by the given Difference D: contrary to the Hypotheſis.

Hence we think it very evident, that Sir *Isaac* neither has demonstrated, nor intended by this Lemma to demonstrate, that any Moment of Time was assignable, wherein these varying Quantities would become *actually equal*, or the Ratios *really the same*; but only that no Difference could be named, which they should not pass. It is certain whenever the Quantities or Ratios compared in this Lemma, are capable of an actual Equality, they must really become so. But when they are incapable of such Equality, the Phrase of *ultimately equal* must of Necessity be interpreted in a somewhat laxer Sense; *i. e.* as Sir *Isaac* in the 71st Proposition of the first Book of his Principia expresses it, *pro æqualibus habeantur*, *are to be esteemed equal*, and means only that such Quantities or Ratios *approach without Limit*. Accordingly we find, that immediately after this Lemma he uses the Expressions *ultimo in ratione æqualitatis*, and *ultimo æquales*, as synonymous Terms. However, as in every Subject of this Lemma, all ultimate Difference is excluded, the Consequences drawn from it, are equally just and perspicuous, whether the Quantities do or do not become *actually equal*, and the Ratios *actually coincident*. And this Restriction of the Sense of this Lemma, is absolutely necessary to be attended to in this Doctrine, because Sir *Isaac* himself has applied it to Quantities and Ratios incapable of an *actual* Equality or Agreement.

If it should be alledged by any, that notwithstanding the Demonstration above, yet the Quantities and Ratios mentioned in the Lemma may differ at last; altho' that Difference be less than any given or assignable Difference: Those Persons would do well to consider, that such a way of reasoning being admitted, would overturn some of the finest Demonstrations of the most accurate Geometricians among the Antients themselves, *viz. Euclid* and

Archimedes,

Archimedes, whose Works have undergone the strictest Scrutiny and Examination of the best Geometricians, since their Time to this very Day. For Example, How does *Archimedes* demonstrate, that a Circle is equal to a rectangular Triangle, having its Base equal to the Circumference, and its perpendicular Altitude equal to the Radius of the Circle? He does it by shewing that a Circle can neither be greater nor less than such a Triangle. But how does he prove this? By shewing that the Circle is neither greater nor less by any given Space. Again, *Euclid* demonstrates that Circles are to each other as the Squares of their Diameters, by shewing that the Square of the Diameter of the one Circle, is to the Square of the Diameter of the other, neither as the first Circle is to a Space greater, nor yet to a Space less than the other Circles. But let us see what he means by a Space greater or less than the other Circle. Why, he means a Space differing from it by a given or assignable Difference; *i. e.* according to the Lemma premised to 2. E. 12; such a Difference as, repeated a certain Number of Times, may exceed that Circle, as appears to any one who reads that Proposition and Lemma; which Lemma is the Foundation of the Method of Exhaustions, made Use of by *Euclid* and *Archimedes* in these and many other Propositions. If any one should now object, that notwithstanding what *Euclid* and *Archimedes* have demonstrated in these Propositions, the Circle may be greater or less than the rectangular Triangle; and the Ratio of the Squares of the Diameters may be greater or less than the Ratio of the Circles, altho' not by any given or assigned Difference, yet by a Difference less than any given Difference: Is not this the very same Objection raised against Sir *Isaac*'s Lemma? And therefore, if it be of no Weight against *Euclid* and *Archimedes*, no more is it against him. But the Truth of the Matter is,

 that

that a Difference less than any Thing assignable, is the same Thing as no Difference at all: For repeat it as often as you please, it can never be equal to any finite Quantity; and therefore can bear no Ratio to it, by *Def.* 4. E. 5. consequently it can be of no Importance to make the Thing greater or less. This Difficulty being removed, let us proceed in our Explication of the Doctrine.

DEF. 1. In this Method, any fixed Quantity, which some varying Quantity, by a continual Augmentation or Diminution, shall perpetually approach, but never pass, is *considered as* the Limit, to which the varying Quantity will at last or ultimately become equal; provided the varying Quantity can be made in its Approach to the other to differ from it by less than by any Quantity how minute soever that can be assigned.

DEF. 2. Ratios also may so vary, as to be confined after the same Manner to some determined Limit, and such Limit of any Ratio is here *considered as* that, which the varying Ratio can approach with any Degree of Nearness, and with which it will ultimately coincide.

More largely a prime or ultimate Ratio may be this defined, *viz.* If there are two Quantities, one or both of which are continually varying, either by being constantly augmented, or diminished; and if the Proportion they bear to each other, does by this Means perpetually vary, but in such a Manner that it constantly approaches nearer and nearer to some determined Proportion, and can also be brought at last in its Approach nearer to this determined Proportion, than to any other that can be assigned, but can never pass it: This determined Proportion is then called the ultimate Proportion, or the ultimate Ratio of those varying Quantities.

From

From any Ratios having ſuch a Limit, it does not follow, that the variable Quantities exhibiting that Ratio have any final Magnitude, or even Limit, which they cannot paſs.

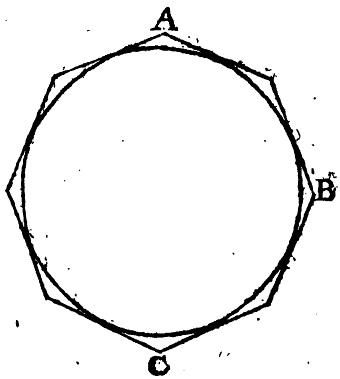
For ſuppoſe two Magnitudes B and B+A, whoſe Difference ſhall be A, are each of them perpetually increaſing by equal Degrees. It is evident from the Nature of Proportion, that if A remains unchanged, the Proportion of B+A to B is a Proportion, that tends nearer and nearer to the Proportion of Equality, as B becomes larger; it is alſo evident, that the Proportion of B+A to B may, by taking B of a ſufficient Magnitude, be brought at laſt nearer to the Proportion of Equality, than to any other aſſignable Proportion; and conſequently the Ratio of Equality is to be conſidered as the ultimate Ratio of B+A to B. The ultimate Proportion then of theſe Quantities is here aſſigned, tho' the Quantities themſelves have no final Magnitude. The ſame holds true in decreaſing Quantities.

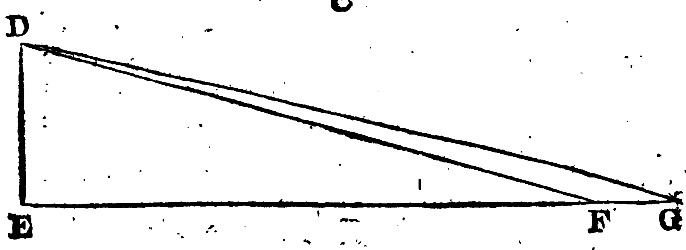
According to the firſt of theſe Definitions, a Circle is to be called the ultimate Magnitude of the Polygon circumſcribing it; becauſe this Polygon, by increaſing the Number of its Sides, can be made to differ from the Circle, leſs than by any Space that can be propoſed how ſmall ſoever; and yet the Polygon can never become actually equal to the Circle, nor leſs.

In like Manner the Circle will be the ultimate Magnitude of the Polygon inſcribed, with this Difference only, that as in the firſt Caſe the varying Magnitude is always greater, here it will be leſs than the ultimate Magnitude, which is its Limit.

Suppoſe EG=the Sum of the Sides of a Polygon circumſcribed, and DE=the $\frac{1}{2}$ Diameter of the Circle, and EF= the Circumference of the Circle; then is the Triangle DEF the ultimate Magnitude of the Tringle DEG; becauſe the Baſe EG being always

always equal to the Circumference of the Polygon, will constantly be greater than the Base EF, equal to





the Circumference of the Circle only, and yet EG may be made to approach EF nearer than by any Difference that can be named.

Upon this first Definition and Explication we may found the following Proposition, *viz.* That when varying Magnitudes keep constantly the same Proportion to each other, their ultimate Magnitudes are in the same Proportion.

This is almost self-evident, for if they preserve their Proportion all the Time they are moving to their fixed Limit, without doubt they have it when they do come there.

Hence we may easily infer the Equality between the Circle and Triangle DEF; for the Circle being the ultimate Magnitude of the Polygon; and the

Triangle DEF the ultimate Magnitude of the Triangle DEG; since the Polygon and the Triangle DEG are equal, by this Proposition, the Circle and Triangle DEF will also be equal. For the ultimate Magnitudes of the same or equal varying Magnitudes are equal.

From the 2d Definition above we may infer, That all the ultimate Ratios of the same varying Ratio are the same with each other.

Suppose the Ratio of A to B continually varies, by the Variation of one or both of the Terms A and B, if the Ratio of C to D be the ultimate Ratio of A to B; and the Ratio of E to F be likewise the ultimate Ratio of the same; then we say, the Ratio of C to D is the same with the Ratio of E to F.

The Truth of this is evident by 11. E. 5.

It now remains that we take some Notice of the proper Meaning of those Words, *nascent* and *evanescent* Quantities, which of late have been disputed. Sir *Isaac* having first compared such Augments as have a finite, that is, a real Magnitude, and found their Proportions; then he is willing to know what that Proportion will be in a particular Case, *viz.* when they are at the End of their vanishing State; and for that Purpose, he supposes these Augments continually to diminish; and having determined the nearest Proportion to which they constantly tend during their Diminution, assigns this as the true Proportion of the Velocities or Fluxions.

Since therefore these vanishing Quantities are expressly declared in the Words above quoted from Sir *Isaac* to be finite and variable; his Expression must be understood to relate to the whole Time they are vanishing.

And his Words are free from any Impropriety; for the Term *vanishing* is daily applied to Objects during the Time of their disappearing, before they are actually out of Sight, absolutely signifying no more

more than going to vanish. Just as we say the Sun is setting, in the most limited Signification of that Word, as soon as its lower Limb touches the Horizon, and as soon as ever the Sun is quite out of Sight, it is no longer setting, but actually set: So these Quantities being of a finite, that is, a real Magnitude, do not vanish instantaneously, but with the utmost Propriety may be said to be vanishing all the Time they are undergoing the Diminution ascribed to them.

It has been objected, that these *vanishing Quantities* are utterly impossible, inconceivable, and utterly unintelligible, and would have it thought, that the Conclusions derived by their Means, must be precarious at least, if not erroneous and impossible. These Objectors ought to consider, that the Symbol *o*, by which these Quantities are generally denoted, at first represents a finite and ordinary Quantity, which must be understood to diminish continually, and as it were by local Motion, till after some certain Time, it is quite exhausted, and terminates in mere Nothing. In its Approach towards Nothing, and just before it becomes absolute Nothing, or is quite exhausted, it must necessarily pass through a Multitude of varying Proportions; for it cannot pass from being an assignable Quantity to Nothing at once; that were to proceed *per saltum*, and not continually, which is contrary to the Supposition. While it is an assignable Quantity, tho' ever so little, it is not yet the exact Truth, in geometrical Rigor, but only an Approximation to it; and to be accurately true, it must be less than any assignable Quantity whatever, that is, it must be a vanishing Quantity. Therefore the Conception of a vanishing Quantity must be admitted as a rational Notion, and intelligible.

If the Impossibility above objected was granted (which we deny) yet would not the Argumentation

be

be at all affected thereby, or the Conclusion the less certain. The Impossibility of Conception may arise from the Narrowness and Imperfection of our Faculties, and not from any Inconsistency in the Nature of the Thing; so that we need not be very solicitous about the positive Nature of these Quantities, or the Names they are called by, but we may confine ourselves wholly to the Use of them, and to discover their Properties. They are not introduced for their own Sakes, but only as so many intermediate Steps, to bring us to the Knowledge of other Quantities, which are real, intelligible, and required to be known. It will appear by the Sequel, that Fluxions will be used in the same Manner as Scaffolding is to a Building; they will assist in raising a Structure, but before it is quite finished, they will be duly eliminated and taken away. The 3d Part of our Design, *viz.* the Application of Fluxions, and the further Prosecution of this Subject must be deferred to our next Number.

To be continued.

CONIC SECTIONS.

The Properties of the ELLIPSE *continued.*

COROLLARY *to* PROP. LI.

THE conjugate Axe, continued from the Center to the focal Tangent, is equal to the Semi-transverse Axe; *that is*, CL(=KE)=CB.

PROPOSITION LII.

If Perpendiculars be drawn from the Vertices to the focal Tangent, then these Perpendiculars shall be equal to the Distance (in the Axe) from each Vertex, to its adjacent Focus respectively; *that is*, AO=AK, and BQ=BK. (*See the Fig. Page* 152.)

DEMONSTRATION.

By the 24, AO×BQ=AK×KB, therefore AO : KA : : KB : BQ; but AO=AK (*by* 51) therefore KB=BQ. Q. E. D.

PROPOSITION LIII.

If, from the Point of Contact of the focal Tangent, a right Line be drawn to the Vertex, and any

any Ordinate be produced to the Tangent to cut that Line; then, the Diſtance, between the Tangent and Interſection of theſe Lines, is equal to the Diſtance (in the Axe) from the Focus to the Application of the Ordinate; *that is*, DN=KV.

DEMONSTRATION.

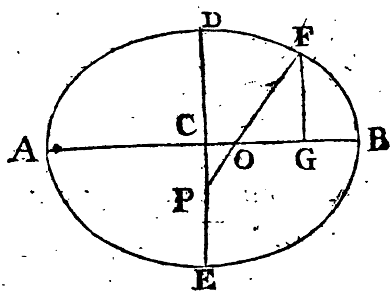
The Triangle LDN is ſimilar to the Triangle LAO, therefore OA : DN : : (LO : LN : :) KA : KV; but (*by* 51.) AO=AK, therefore DN=KV. Q. E. D.

PROPOSITION LIV.

If, from any Point (P) of the conjugate Axe, a right Line PO, equal to the Difference of the Semi-tranſverſe and Semi-conjugate, be applied to the tranſverſe Axe, and from thence continued, ſo that the external Part OF be equal to the Semi-conjugate Axe; then, I ſay, the Extremity F, of that Line, ſhall be in the Curve of the Ellipſe.

DEMONSTRATION.

Let CO=b, OG=d, CG=x $(b+d)$ and the other Symbols as uſual; then PO=$\frac{1}{2}t-\frac{1}{2}c$, OF=$\frac{1}{2}c$, and

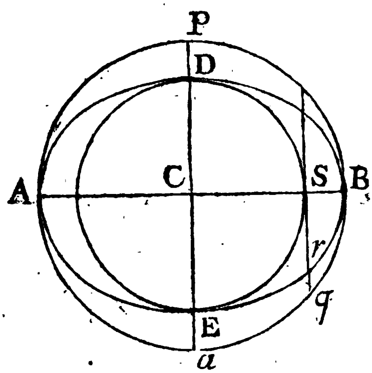


(*by ſimilar Triangles*) $b : d : : \frac{1}{2}t-\frac{1}{2}c : \frac{1}{2}c$; whence (*by Compoſition*) x $(b+d) : d : : \frac{1}{2}t : \frac{1}{2}c$, and $x^2 : d^2 : : \frac{t^2}{4} : \frac{c^2}{4}$; therefore $\frac{\frac{1}{4}c^2 \times x^2}{\frac{1}{4}t^2} = d^2 =$ (*by* 47. *E.* 1.)

$\frac{1}{4}c^2 - y^2$, consequently $y^2 = \left(\frac{1}{4}c^2 - \frac{\frac{1}{4}c^2 \times x^2}{\frac{1}{4}t^2} = \right)$ $\frac{\overline{\frac{1}{4}t^2 - x^2} \times \frac{1}{4}c^2}{\frac{1}{4}t^2} = \frac{\overline{\frac{1}{2}t + x} \times \overline{\frac{1}{2}t - x} \times \frac{1}{4}c^2}{\frac{1}{4}t^2}$, and from thence $\frac{1}{4}t^2 : \frac{1}{4}c^2 :: \overline{\frac{1}{2}t + x} \times \overline{\frac{1}{2}t - x} : y^2$; *that is*, $AC^2 : CD^2 :: AG \times GB : GF^2$. Q. E. D.

PROPOSITION LV.

If a Circle be described on the transverse Axe of the Ellipse, and Ordinates be drawn to both Curves; it will be, as the transverse Axe is to the Conjugate, so is any Ordinate in the Circle, to its corresponding Ordinate in the Ellipse; *that is*, $AB : DE :: sq : sr$.



DEMONSTRATION.

By 1st. $AB^2 : DE^2 :: \overline{sq})^2 \,(= by\ 35.\ E.\ 3.\ As \times sB)$ $: \overline{sr})^2$; therefore $AB : DE :: sq : sr$. Q. E. D.

PROPOSITION LVI.

As the transverse Axe, is to the conjugate Axe, so is the Area of a Circle on the transverse Axe, to the Area of the Ellipse.

DEMON-

DEMONSRATION.

By the preceding, $t : c :: sq : sr ::$ (*by* 12. *E.* 5.) all the sq's : all the sr's :: the Circle described on t : the Ellipsis.

PROPOSITION LVII.

The Area of every Ellipse is equal to the Area of a Circle, whose Diameter is a Line equal to the Square Root of the Rectangle of the transverse Axe into the Conjugate.

DEMONSTRATION.

By the preceding, the Circle described on t : the Ellipsis $:: t : c :: t^2 : ct ::$ (*by* 1. *E.* 12.) the Circle described on t : to that described on a Line $= \sqrt{tc}$; therefore the Ellipsis = the Circle described on a Line $= \sqrt{tc}$. Q. E. D.

COROLLARY.

Since the Circle on t : Ellipsis $:: t^2 : tc$, it appears that the Areas of Circles are to those of Ellipses, as the Square of the Diameters of Circles to the Rectangle under the transverse and conjugate Axes of the Ellipses.

PROPOSITION LVIII.

Every Ellipse is a Mean-proportional between the Circle on its transverse and that on its conjugate Axe.

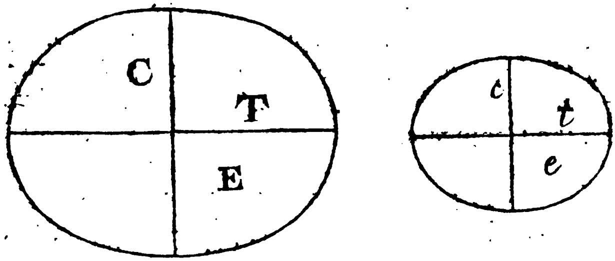
DEMONSTRATION.

By 56, the Circle on t : Ellipsis $:: t : c :: tc : c^2 ::$ (*by* 2. *E.* 12.) the Circle on a Line $= \sqrt{tc}$: the Circle on $c ::$ (*by preced.*) Ellipsis : Circle on c. Q. E. D.

PRO-

PROPOSITION LIX.

Ellipses are to each other, in a Ratio, compounded of the subduplicate Ratio of their Parameters, and sesquiplicate Ratio of their transverse Axes directly.



DEMONSTRATION.

By the 57th. the Ellipsis $E=$ the Circle on $\sqrt{TC}$, and the Ellipsis $e=$ the Circle on $\sqrt{tc}$, therefore $E : e ::$ the Circle on $\sqrt{TC}$: the Circle on $\sqrt{tc} ::$ (*by* 2. *E.* 12.) $TC : tc ::$ (*because* $c=\sqrt{tp}$, *and* $C=\sqrt{TP}$) $T^{\frac{3}{2}}\times P^{\frac{1}{2}} : t^{\frac{3}{2}}\times p^{\frac{1}{2}}$. Q. E. D.

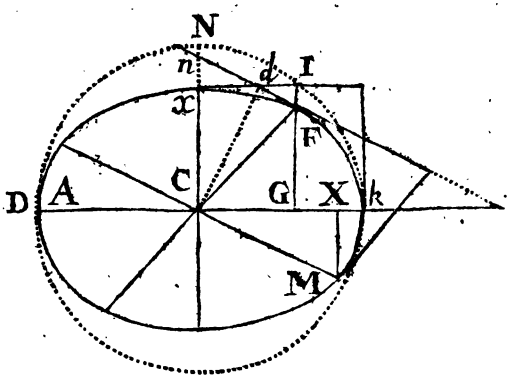
PROPOSITION LX.

Parallelograms, drawn with their Sides parallel to the conjugate Diameters, and circumscribing the Ellipse, are equal.

DEMONSTRATION.

On the transverse Axe describe the Circle *k*ND; continue the Ordinate through the Point of Contact to I; draw the Ordinate MX, and C*d* perpendicular to the Tangent; then, I say, $Cd\times CM=Cx\times Ck$. For let GI= (*by* 48.) CX be put$=b$, $Cn=d$, $CM=D$, $Cd=p$, $GF=y$, $Cx=c$ and $CK=t$; then (*by*

(*by Prop.* 55.) $y : b :: c : t$, therefore $y = \frac{bc}{t}$; but (*by* 39*th*) $\frac{bc}{t}$ $(y) : c :: c : d$ and (*by Sim. Trian.*) b :



D : : $p : d$; whence, by comparing the two last Proportions, we have $\frac{ct}{b}$ $(=d) = \frac{Dp}{b}$, or $ct = Dp$; *that is*, $Cd \times CM = Cx \times Ck$. Q. E. D.

Proposition LXI.

As the Distance between the Foci, is to the transverse Axe, so is the Distance between the Focus and the Vertex, to the Distance between the Vertex and Intersection of the focal Tangent with the Axe produced; *that is*, KH : AB : : BK : : BT. *See Fig. to Prop.* 51.

Demonstration.

By the 17th. AK : BK : : AT : BT, therefore AK— (BK) AH : BK : : AT—BT : BT; that is KH : BK : : AB : BT. Q. E. D.

Proposition LXII.

If a right Line be drawn from the Focus to any Point of the Curve, and, from that Point, a Line be

be drawn parallel to the Axe, and continued to the Perpendicular which cuts the Axe produced in the Point of Interfection of the focal Tangent; then these Lines are in the conftant Ratio of the Diftance between the Foci to the tranfverfe Axe; *that is*, KE : E*n* : : KH : AB. *See Fig. to Prop.* 51.

DEMONSTRATION.

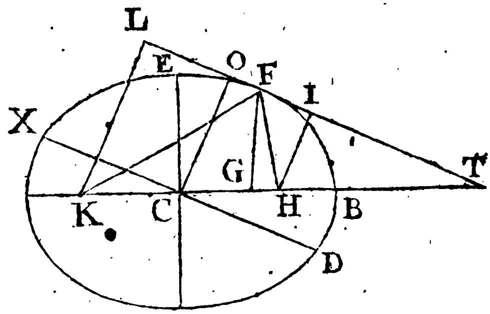
By Prop. 51. CL=KE, and (*by* 52.) BQ=BK; therefore (*by fim. Trian.*) CL : CT : : BQ : BT; *that is*, KE : E*n* : : BK : BT : : (*by the preced.*) HK : AB. Q. E. D.

PROPOSITION LXIII.

The focal Diftance of any Point in the Curve, is to a Perpendicular let fall from that Focus to the Tangent of the faid Point, as the Semi-conjugate Diameter, to the Semi-conjugate Axis.

DEMONSTRATION.

The Triangles FHI, FKL (*by Prop.* 27.) are fimilar, therefore HF : FK : : HI : LK, whence (*by*



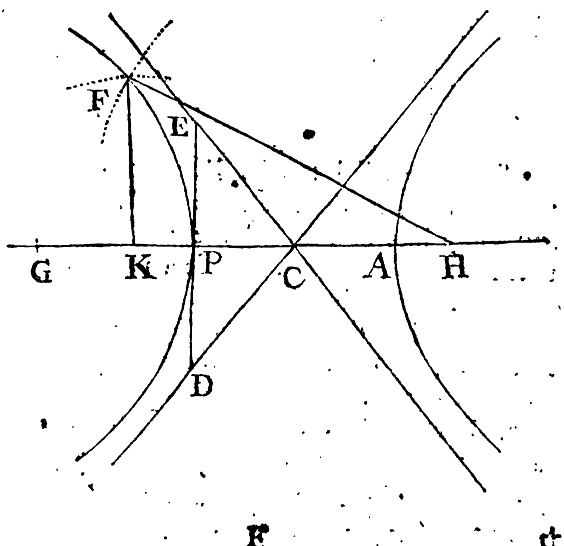
Compofition and Alternation,) HF+FK (2BC) : HI+LK (2CO) : : BC : CO : : FH : HI : : BC×CD : CO×CD; but (*by Prop.* 60.) OC×CD=BC×CE, therefore FH : HI : : BC×CD : BC×CE : : CD : CE. Q. E. D.

CONIC SECTIONS: PART III.

Of the HYPERBOLA.

The GENESIS.

UPON a Plain, take any ſtraight Line AP, in which, continued both Ways, make PK=AH and let the Point G be taken any where (without H and K) in that Line; then, if, with the Radius AG from the Point H, as a Center, you deſcribe an Arc; and from the Center K, with



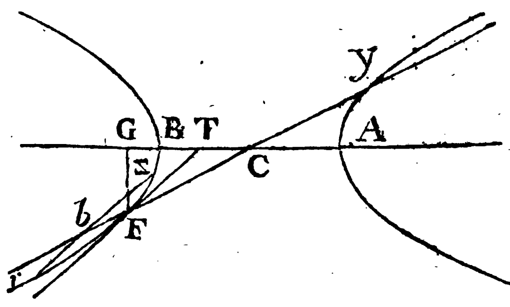
the Radius PG, you intersect the former Arc at F; also, if, from the Points H and K, you draw the Lines HF, and FK, I say, HF—FK=AP.

For, by Construction, HF (AG)=AP+PG, and FK=PG; therefore HF—FK=(AP+PG—PG=) AP. In like Manner, an indefinite Number of Points may be found; and the curve Line drawn through them all is called an Hyperbola.

DEFINITIONS.

1. The Points H and K are called the focal Points, or Foci.

2. A Diameter of the Hyperbola is a right Line which passes through C, the Middle of AB, and being produced bisects all the Lines within the Curve



which are parallel to the Tangent drawn through the Point where the Diameter intersects the Curve, and the Lines so bisected are called Ordinates to that Diameter. Thus, FY is a Diameter, and *rb*, *bz* are Ordinates being parallel to the Tangent FT, which touches the Curve in F the Vertex of the Diameter, or the Point where the Diameter intersects the Curve.

3. The Point of Concourse of all the Diameters (as C) is called the Center.

4. That produced Diameter to which the Ordinates stand at right Angles (as AB) is called the Axe.

5. The

5. The common Interſection of the Diameter produced and the Ordinate (as G, or b) is called the Point of Application.

6. That Part of the Diameter produced, which is intercepted between the Vertex and Point of Application, is called the Abſciſſa, as BG or Fb.

7. If, on (P) the Vertex of the Axe, a Perpendicular to the Axe be drawn and continued both Ways *(See the Fig. to the Geneſis)* and then, if, from the Center C, with the Radius CK, you interſect that Perpendicular in the Points D and E, Right-lines drawn from the Point C through E and D are called Aſſymptotes; and the Perpendicular intercepted between them (as ED) is called the conjugate Axe.

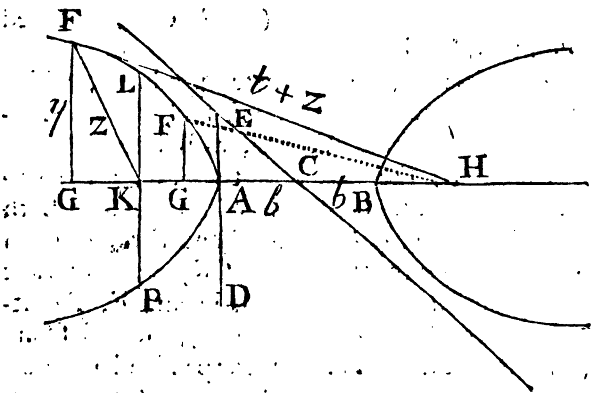
PROPOSITION I.

As the Square of the tranſverſe Axe, is to the Square of the conjugate Axe, ſo is the Rectangle of the Abſciſſa into the Sum of the Tranſverſe and Abſciſſa, to the Square of the Ordinate applied to that Abſciſſa; *that is,* $AB^2 : DE^2 :: BG \times AG : GF^2$.

DEMONSTRATION.

Put $AC = \frac{1}{2}t$, $AE = \frac{1}{2}c$, $CG = x$, $CK = CH = b$, $GF = y$, and $FK = z$; then $GK = b - x$ or $x - b$ according as the Point G falls on this or that Side the Focus and $KH = 2b$; alſo *(by the Geneſis)* $FH = t + z$, and $AE^2 + AC^2$ $(CE^2) = CK^2$, *that is,* $\frac{1}{4}c^2 + \frac{1}{4}t^2 =$ *(by 47. E. 1.)* b^2; and $HF^2 =$ *(by 12. and 13. E. 2.)* $KH^2 + FK^2 \pm 2KH \times GK$, *that is,* $t^2 + 2tz + z^2 = z^2 + 4b^2 + 4bx - 4b^2$; therefore $z = \frac{4bx - t^2}{2t}$ and by ſquaring both Sides $\frac{16b^2x^2 - 8bt^2x + t^4}{4t^2} = (z^2 =) y^2 + x^2 - 2bx + b^2$, which reduced gives $16b^2x^2 + t^4 = 4t^2y^2 + 4t^2x^2 + 4t^2b^2$: Now if in this Equation, for $16b^2$ and $4b^2$,

 we



we substitute their respective Values found from the first, we shall have $t^2y^2=c^2x^2-\frac{1}{4}t^2c^2$; which converted to an Anolagy gives $t^2 : c^2 :: \overline{x+\frac{1}{2}t}\times\overline{x-\frac{1}{2}t} : y^2$, or $AB^2 :: DE^2 :: BG\times AG : FG^2$. Q. E. D.

COROLLARY.

Let the transverse and conjugate Axes be represented by t and c, any Abscissa and its Ordinate by x and y, then by this Theorem $t^2 : c^2 :: \overline{t+x}\times x : y^2$, therefore $t^2y^2=c^2tx+c^2x^2$, which is the Equation of the Curve.

Definition. A third Proportional to the transverse and conjugate Axe, is called the Parameter of the Axe; *that is*, if p be put for the Parameter $t : c :: c : p$, therefore $tp=c^2$.

PROPOSITION II.

As the transverse Axe is to the Parameter of the Axe, so is the Rectangle of the Abscissa into the Sum of the Transverse and Abscissa, to the Square of the Ordinate applied to that Abscissa; *that is*, $t : p :: \overline{t+x}\times x : y^2$.

DEMON-

DEMONSTRATION.

By the preceding Definition $tp=c^2$; therefore if, in the Equation of the Curve, tp be substituted for c^2, we shall have $ty^2=tpx+px^2$ (which is the Equation of the Curve in the Terms of the Parameter) which being put in an Anolagy gives $t:p::\overline{t+x}\times x:y^2$. Q. E. D.

COROLLARY.

Hence it appears, that the Rectangle of any Abscissa into the Sum of the Transverse and the said Abscissa, is to the Square of the Ordinate applied to that Abscissa, as the Rectangle of any other Abscissa into the Sum of the Transverse and that Abscissa, to the Square of the Ordinate applied to that Abscissa: For (*by the Prop.*) $\overline{t+x}\times x:y^2::t:p::\overline{t+X}\times X:Y^2$.

PROPOSITION III.

As half the transverse Axe, is to the Sum of the Transverse and focal Distance, so is the focal Distance, to half the Parameter of the Axe; *that is*, (by putting q for the focal Distance) $\frac{1}{2}t:\frac{1}{2}t+q::q:\frac{1}{2}p$.

DEMONSTRATION.

CK (CE)−CA=AK, *that is* $\sqrt{\frac{1}{4}t^2+\frac{1}{4}c^2}-\frac{1}{2}t=q$; but (*by the preced.*) $\frac{1}{4}c^2=\frac{1}{4}pt$, therefore $\sqrt{\frac{1}{4}t^2+\frac{1}{4}tp}-\frac{1}{2}t=q$, and $\frac{1}{4}tp=tq+q^2$; *that is*, $\frac{1}{2}t:t+q::q:\frac{1}{2}p$. Q. E. D.

PROPOSITION IV.

The Parameter of the Axe is equal to double the Ordinate passing tho' the Focus; *that is* (if y be put for the Ordinate passing thro' the Focus) $y=\frac{1}{2}p$, or $p=2y$.

DEMON-

Demonstration.

By the 2d (putting q for the focal Distance) $t : p :: \overline{t+q} \times q : y^2$, and (*by the preced.*) $\overline{t+q} \times q = \frac{1}{4}tp$; therefore, by Substitution, $t : p :: \frac{1}{4}tp : y^2 = \frac{1}{4}p^2$, whence $y = \frac{1}{2}p$. Q. E. D.

Proposition V.

As the Sum of the transverse Axe and its Parameter, is to the Distance between the Foci, so is the Distance between the Foci, to the transverse Axe.

Demonstration.

Let $KH = b$, then $\frac{1}{2}b = (\frac{1}{2}KH = CK = CE)$ $\sqrt{\frac{1}{4}t^2 + \frac{1}{4}c^2}$; and $\frac{1}{4}b^2 = \frac{1}{4}t^2 + \frac{1}{4}c^2$, or $b^2 = t^2 + c^2$. But (*by Prop.* 2.) $tp = c^2$, therefore $b^2 = t^2 + tp$, whence $t + p : b :: b : t$. Q. E. D.

Proposition VI.

A Fourth-proportional to the conjugate Axe, transverse Axe, and any Ordinate, is a Mean-proportional between the Abscissa of that Ordinate and the Sum of the Transverse and Abscissa.

Demonstration.

Let the Fourth-proportional be b; then $c : t :: y : b$, therefore $\frac{ty}{c} = b$. But (*by Prop.* 1.) $t^2 : c^2 :: \overline{t+x} \times x : y^2$, or $t : c :: \sqrt{\overline{t+x} \times x} : y$, therefore $\sqrt{\overline{t+x} \times x} = \frac{ty}{c} = b$. Q. E. D.

Proposition VII.

As the Square of any Ordinate, is to the Rectangle of the Abscissa into the Sum of the Transverse and

and Abſciſſa, ſo is the Square of the conjugate Axe, to the Difference between the Square of the conjugate Axe and that of the Diſtance of the Foci.

DEMONSTRATION.

Let the Diſtance between the Foci be $=b$, then $c^2+t^2=b^2$, whence $t^2=b^2-c^2$. But (*by the 1ſt.*) $y^2 : \overline{t+x}\times x :: c^2 : b^2-c^2 (t^2)$. Q. E. D.

PROPOSITION VIII.

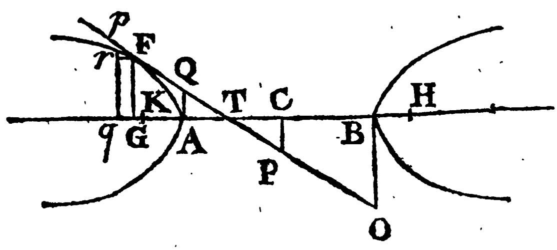
As the Square of any Ordinate, is to the Rectangle of the Parameter of the Axe into the Abſciſſa, ſo is the Sum of the ſaid Rectangle and the Square of the conjugate Axe, to the Square of the conjugate Axe; *that is*, $y^2 : px :: c^2+px : c^2$.

DEMONSTRATION.

By the Equation of the Curve $t^2y^2=c^2tx+c^2x^2$, and (*by the 2d*) $\frac{c^2}{p}=t$; therefore (by Subſtitution, &c.) $c^2y^2=pxc^2+p^2x^2$, whence $y^2 : px :: c^2+px : c^2$. Q. E. D.

PROPOSITION IX.

As the Diſtance from the Center to the Ordinate drawn from the Point of Contact of any Tangent, is to the Abſciſſa of that Ordinate, ſo is the Sum of the Tranſverſe and Abſciſſa, to the Subtangent; *that is*, CG : AG :: BG : GT.



DE-

DEMONSTRATION.

Suppose Fp an indefinitely small Part of the Curve, and produced so as to cut the Axe in T; draw the Ordinate FG and pq parallel thereto; also draw Fr parallel to the Axe, and put AT$=a$, F$r=q$G$=n$, and $rp=m$; then GT$=a+x$, B$q=t+x+n$, A$q=x+n$, and $pq=y+m$. Now, $pr : r$F :: FG : GT, *that is*, $m : n :: y : x+a$, therefore $\frac{ny}{m}=x+a$; moreover (*by Prop.* 2.) $t : p :: \overline{t+x+n}\times\overline{x+n} : \overline{y+m}\times\overline{y+m}$, and $t : p :: \overline{t+x}\times x : y^2$; whence (*by the first Analogy*) $ptx+ptn+px^2+2pxn-2tym=ty^2=$ (*by the 2d Anal.*) $ptx+px^2$; therefore $ptn+2pxn=2tym$, and $n=\frac{2tym}{pt+2px}$. But $a+x=n\times\frac{y}{m}$, therefore $a+x=$ $\left(\frac{2tym}{pt+2px}\times\frac{y}{m}=\frac{2ty^2}{pt+2px}=\frac{ty^2}{p}\times\frac{2}{t+2x}=tx+x^2\times\frac{2}{t+2x}=\frac{2tx+2x^2}{t+2x}=\right)\frac{tx+x^2}{\frac{1}{2}t+x}$, which converted to an Analogy gives $\frac{1}{2}t+x : x :: t+x : a+x$, or CG : AG :: BG : GT. Q. E. D.

PROPOSITION X.

The Sum of the Abscissa of the Ordinate, from the Point of Contact, and half the Transverse Axe, is to half the transverse Axe, as half the transverse Axe, to the Distance (in the Axe produced) from the Center to the Intersection of the Tangent; *that is*, CG : CA :: CA : CT.

DEMONSTRATION.

CT$=$CG$-$GT: But CT$=\frac{1}{2}t-a$, CG$=\frac{1}{2}t+x$, and (by the last) GT$=\frac{tx+x^2}{\frac{1}{2}t+x}$; therefore $\frac{1}{2}t-a=$

($\frac{1}{2}t+$

$(\frac{1}{2}t+x-\frac{tx+x^2}{\frac{1}{2}t+x}=)\ \frac{\frac{1}{4}t^2}{\frac{1}{2}t+x}$, whence $\frac{1}{2}t+x : \frac{1}{2}t :: \frac{1}{2}t : \frac{1}{2}t-a$, or CG : CA :: CA : CT. Q. E. D.

PROPOSITION XI.

The Sum of the Abſciſſa of the Ordinate, drawn from the Point of Contact, and half the Tranſverſe, is to half the tranſverſe Axe, as the Abſciſſa, to the Diſtance between the Vertex and Interſection of the Tangent; *that is*, CG : AC : : AG : AT.

DEMONSTRATION.

By the preced. $\frac{1}{2}t-a=\frac{\frac{1}{4}t^2}{\frac{1}{2}t+x}$, therefore $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$; whence $\frac{1}{2}t+x : \frac{1}{2}t : : x : a$, or CG : AC : : AG : AT. Q. E. D.

PROPOSITION XII.

The Sum of the Abſciſſa of the Ordinate, from the Point of Contact, and half the Tranſverſe, is to half the Tranſverſe; as the Sum of the Tranſverſe and Abſciſſa, to the Difference between the Tranſverſe and the external Part, *that is*, CG : CA : : BG : BT.

DEMONSTRATION.

By the preced. $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$, therefore $t-a=$ $\left(t-\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}\right)\frac{\frac{1}{2}t^2+\frac{1}{2}tx}{\frac{1}{2}t+x}$; whence $\frac{1}{2}t+x : \frac{1}{2}t : : t+x : t-a$, or CG : CA : : BG : BT. Q. E. D.

PROPOSITION XIII.

The Sum of the Abſciſſa of the Ordinate, from the Point of Contact, and the Tranſverſe, is to the

Difference between the Transverse and the external Part, as the Abscissa, to the external Part; *that is*, BG : BT : : GA : AT.

DEMONSTRATION.

By the 11th. $\frac{1}{2}t+x : \frac{1}{2}t :: x : a$, and (by the preced.) $\frac{1}{2}t+x : \frac{1}{2}t :: t+x : t-a$; therefore (by Equality) $t+x : t-a :: x : a$, or BG : BT : : AG : AT. Q. E. D.

PROPOSITION XIV.

As the Difference between the Transverse and the external Part, is to half the Transverse, so is the external Part, to the Abscissa of the Ordinate from the Point of Contact; *that is*, CT : CA : : AT : AG.

DEMONSTRATION.

By the 11th. $a=\frac{\frac{1}{2}tx}{\frac{1}{2}t+x}$; therefore $x=\frac{\frac{1}{2}ta}{\frac{1}{2}t-a}$, and $\frac{1}{2}t-a : \frac{1}{2}t :: a : x$, or CT : CA : : TA : AG. Q. E. D.

PROPOSITION XV.

As the Difference between the external Part, and half the Transverse, is to the Difference between the Transverse and the external Part, so is the external Part, to the Sum of the external Part and the Abscissa of the Ordinate from the Point of Contact; *that is*, CT : BT : : AT : GT.

DEMONSTRATION.

By the preced. $x=\frac{\frac{1}{2}ta}{\frac{1}{2}t-a}$, therefore $x+a=$ $\left(a+\frac{\frac{1}{2}ta}{\frac{1}{2}t-a}\right)\frac{ta-a^2}{\frac{1}{2}t-a}$; whence $\frac{1}{2}t-a : t-a :: a : x+a$, or CT : BT : : AT : GF. Q. E. D.

ANSWERS

ANSWERS TO THE PROBLEMS

Proposed in the Third NUMBER.

PROBLEM XLI. *Answered by* John Turner.

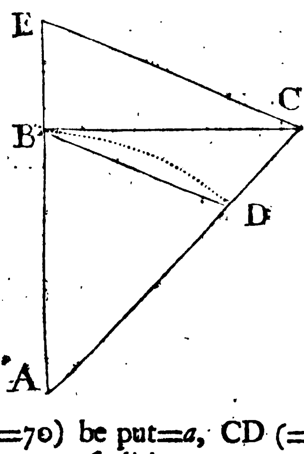
LET the Pence in 308 *l.* 6 *s.* 8 *d*, the Sum of the Notes, be represented by a; the Pence in 8 *l.* 6 *s.* 8 *d.* the given Premium by b; and the Pence in 8 *l.* 11 *s.* 4 *d.* $\frac{2}{3}$, the given Interest, by c: Also let x represent the Pence in the Note due at six Months, and y the Rate of Interest; then $\frac{xy}{2}$ will be the Interest due upon x, and $\frac{2ay-2xy}{3}$ that due upon $a-x$; whence, by the Question, $\frac{xy}{2}+\frac{2ay-2xy}{3}=c$, and therefore $x=\frac{4ay-6c}{y}$. Now $\frac{yx}{2+y}$ is the Premium for discounting x, and $\frac{2ay-2xy}{3+2y}$ that for discounting $a-x$,

 therefore

therefore $\frac{yx}{2+y}+\frac{2ay-2xy}{3+2y}=b$, and $x=\frac{2ay^2+4ay-6b-7by-2by^2}{y}$; whence $\frac{4ay-6c}{y}=\frac{2ay^2+4ay-6b-7by-2by^2}{y}$, and therefore $\overline{2a-2b}\times y^2-7by=6\times\overline{b-c}$, which reduced gives $y=.05$ the Rate of Intereſt required, and therefore $x=49200$; conſequently the Value of the Note due at the End of ſix Months is 205 *l.* and that due at the End of eight Months 103 *l.* 6 *s.* 8 *d.*

PROBLEM XLII, *Anſwered by Mr.* W. Kingſton *of* Bath.

In the right-angled Triangle ABC, let A repreſent the Anchor, B the Place of the Ship in the firſt Poſition, and C her Place in the laſt Poſition, and upon A as a Center let the Arch BD be deſcribed;



then, if BC (=70) be put $=a$, CD (=50) $=b$, and AB (=AD) $=x$, we ſhall have $a^2+x^2=b^2+2bx+x^2$; whence $x=\frac{a^2-b^2}{2b}=24$ the required Depth of the Water.

The

The same answered by Mr. Thomas Moss *of* Deptford.

CONSTRUCTION.

Draw the indefinite right Line AE, in which take EB = (50) the Length of Cable veered out; make BC perpendicular to AE and equal to (70) the Distance between the two Positions of the Ship; then join E,C and draw CA to make an Angle with EC equal to the Angle AEC, and the Thing is done.

DEMONSTRATION.

Draw BD parallel to EC:

Since the Angles E and ECA are equal (*by Constr.*) and BD parallel to EC, it is evident that CA = AE, BA = DA and DC = BE = the Length of Cable veered out; therefore BA = the required Depth of Water. Q. E. D.

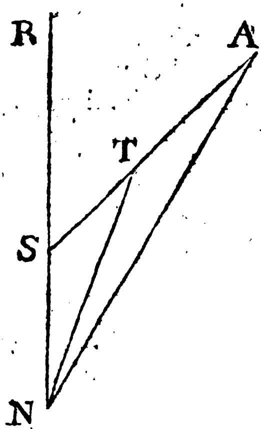
CALCULATION.

In the right-angled Triangle EBC are given the two Sides BE and BC, whence the Angles E and BCE are given $= 54^d. 27'. 35''.$ and $35°. 32'. 25''.$ respectively; and from thence BCA = ECA (E) — BCA will be found: Then, since BE and BA are Tangents of the Angles ECB and BCA to the Radius BC, it will be as Tangent ECB : Tangent BCA : : EB : BA = 24 the required Depth of Water.

PROBLEM XLIII. *Answered by* John Turner.

Let S be the southermost and N the northermost of the two given Ports; make RSA = 45° the Angle which the Ship from the former makes with the Meridian, and let ST (= 20) represent her Distance run before the other Ship set sail: Then in the Triangle STN are given the Sides ST and SN together with their included Angle TSN, whence NT = 38.81 and the Angles STN = $23°. 37'. 48''.$ and SNT

SNT=21°. 22′. 12″. will likewise be given. Then, in the Triangle NTA will be given the Side NT, the Angle T and the Ratio of NA to TA as 5 to 4;



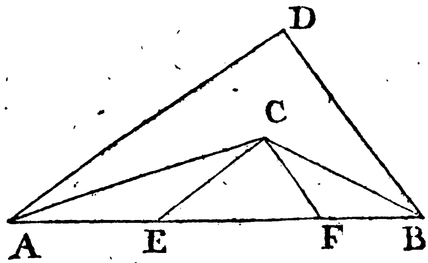
whence the Angle ANT will be found=18°. 42′. 10″, which added to TNS gives SNA=40°. 4′. 22″. the Course required. Moreover, because in the Triangle SAN the Side SN and all the Angles are known, the Distance SA sailed by the first Ship and NA that sailed by the second will be found= 165.27 and 181, 12 respectively.

Problem XLIV. *Answered by Mr.* William Kingston.

Construction.

Upon AB equal to the given Perimeter, let the Triangle ABD be constituted equiangular to that required, and let the two Angles A,B be bisected by Right-lines meeting each other in C; draw CE and CF parallel to DA and DB respectively, then EFC will be the Triangle required.

2 De-



Demonstration.

It is evident (*by Constr.*) that the Triangle EFC is equiangular to ABD. Moreover, the Angle EAC being (=CAD)=ECA, the Side EC will be equal to EA; and for the like Reason FC=FB; therefore EF+EC+CF=EF+EA+FB=AB. Q. E. D.

Method *of* Calculation.

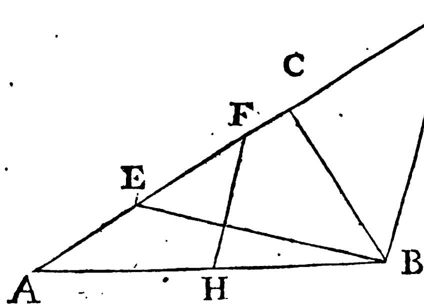
In the Triangle ABC are given all the Angles and the Side AB, whence AC and CB will be known; then, in the isosceles Triangles AEC, BFC are given all the Angles and one Side in each, from which EC and FC will also become known.

Problem XLV. *Answered by* John Turner.

Construction.

Let ABC be the given Triangle, and ABD the Angle which the dividing Line is to make with the Side AB; also let AC be produced to meet BD in D, and let the Ratio of the Part cut off to the whole, be that of AE to AC: Take AF a mean Proportional between AD and AE, draw FH parallel to DB and the Thing is done.

De-



DEMONSTRATION.

Join B, E: Then AFH : ADB : : AF^2 ($AD \times AE$) : AD^2 : : AE : AD : : ABE : ADB; therefore the Consequents being the same the Antecedents AFH, ABE must be equal: But ABE(AFH) : ABC : : AE : AC. Q. E. D.

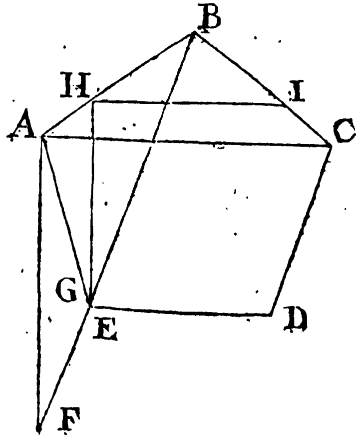
METHOD *of* CALCULATION.

In the Triangle ABD are given all the Angles and the Side AB, whence AD, and consequently AF ($=\sqrt{AE \times AD}$) will be found.

PROBLEM XLVI. *Answered by* John Turner.

CONSTRUCTION.

Join the angular Points A, G of the given Pen-



tagon

tagon ABCDE and draw AF perpendicular and equal to AC; alſo join F, B and from G the Point where BF interſects AE draw GH parallel to FA, which will be a Side of the Square.

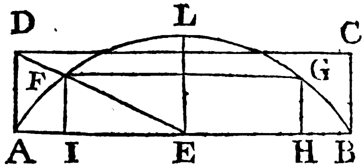
DEMONSTRATION.

Becauſe AF, HG as well as AC and the Side of the Square HI are parallel, the Triangles BAF and BHG as well as BAC and BHI will be ſimilar; therefore BA : AF : : BH : HG; and BA : AC : : BH : HI: But AF and AC are equal by Conſtruction, therefore HG and HI, being Conſequents to the equal Antecedents, muſt likewiſe be equal. Q. E. D.

PROBLEM XLVII. *Anſwered by Mr.* William Kingſton.

CONSTRUCTION.

Let ABL be the given Portion of a Circle, and upon the Chord AB let the Rectangle ABCD be conſtituted whoſe Length is to its Breadth in the given Proportion: Biſect AB in E, and draw DE, and from the Point F where it interſects the Circle draw FG parallel to AB; alſo draw FI and GH parallel to EL, and the Thing is done.



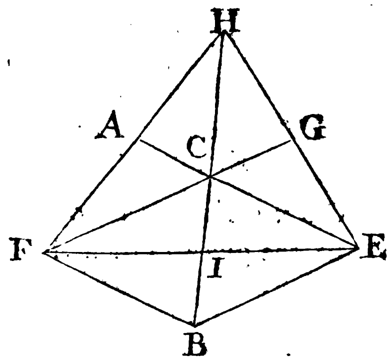
DEMONSTRATION.

Becauſe of the ſimilar Triangles EAD, EIF, it will be AD : IF : : AE : EI : : AB : IH. Q. E. D.

PROBLEM XLVIII. *Answered by* Geometricus.

CONSTRUCTION.

Let a Triangle FCB be constituted, whose Sides are respectively equal to two thirds of the given bisecting Lines, and complete the Parallelogram FE; also draw the Diagonals FE and BC, in the latter of which, produced, take CH=BC; *join* F, H and E, H and FEH will be the Triangle required.



DEMONSTRATION.

Let FC and EC be produced to meet the Sides of the Triangle in G and A:—Since the Diagonals of a Parallelogram bisect each other, EI is therefore equal to FI and CI=$\frac{1}{2}$BC=$\frac{1}{2}$CH; whence HI (bisecting FE) is equal to $\frac{3}{2}$BC, one of the given Lines by Construction: Moreover, the Triangles HBF and HCA being similar and HC=$\frac{1}{2}$HB *(by Constr.)* it follows that HA is=AF, and that CA =$\frac{1}{2}$BF, and therefore EA (bisecting HF) =EC+CA=BF+CA=$\frac{3}{2}$BF. In the very same Manner it will appear that EG is=GH, and that FG is =$\frac{3}{2}$FC. Q. E. D.

METHOD

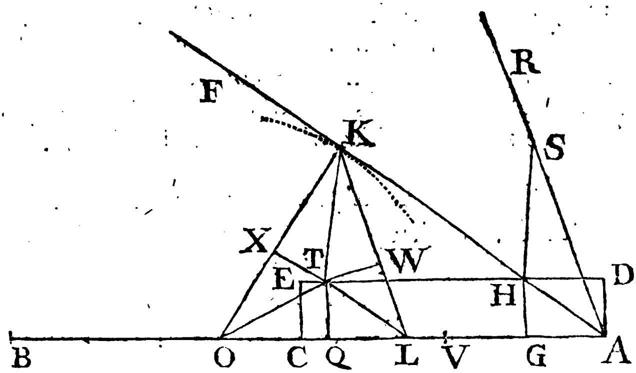
METHOD *of* CALCULATION.

In the Triangle FBC, all the Sides being known, the Angles will be found; then in the Triangle ICF, besides the Angle C, will be given the Side FC, and CI $= \frac{1}{2}$BC, whence FI and the Angle I will be known; whence (HI being given) FH and HE will be found.

PROBLEM XLIX. *Answered by* John Turner.

CONSTRUCTION.

Bisect AB the given Perimeter in C, and upon AC constitute the Rectangle AE equal to the Area of the required Triangle; make CAF equal to half the given Angle, and from H where AF cuts DE let fall the Perpendicular HG; take GV equal to GA and bisect BV in O, then with OB (as a Radius) describe an Arc cutting AF in K; join O, K and make AKL equal to OAF, then OKL will be the Triangle required.



DEMONSTRATION.

Bisect the Angle OKL with the Line KT meeting DE (produced if needful) in T; also draw AR parallel to LK, and HS, parallel to TK, meeting AR in S; then join L, T.

Because the Angle $AKL = OAK = \frac{1}{2}$ the given Angle (*by Constr.*) OLK and consequently OAR will be equal to the whole given Angle; whence $AL = KL$ and consequently $OK\,(OB) + OL + LK\,(AL) = AB$ the given Perimeter. Moreover, since the Lines TK, HS and LK, AS are parallel, we shall have $KT = SH$, and $LK = AS$, also the Angle $TKL = HSA$, and $KLT = SAH = OAK = \frac{1}{2}$ the given Angle (*by Constr.*) whence LT bisects the Angle OLK. Now, if from T the several Perpendiculars TQ, TW, TX be let fall, and the Line TO be drawn, it is obvious that the Area of the Triangle OKL will be equal to $TQ \times \overline{\frac{1}{2}OL + \frac{1}{2}LK + \frac{1}{2}OK} = TQ \times \frac{1}{2}AB = CE \times CA = CD =$ the given Area by Construction. W. W. D.

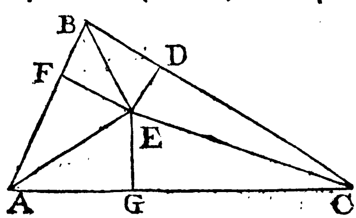
Method *of* Calculation.

In the Triangle AHG are given all the Angles and the Side HG, whence $AG = GV$ is found, and consequently $BV = 2BO = AB - AV$. Then in the Triangle AOK are given the Sides OK (OB) and OA, together with the Angle OAK, whence the Angle OKA and consequently the Angles OKE and KOL may be found. Moreover, in the Triangle OKL all the Angles and the Side OK are known, whence the other Sides will likewise be known.

Problem L. *Answered by* John Turner.

It is evident that

$$AF^2 + FE^2 = (AE^2 =)\ AG^2 + GE^2;$$
$$GC^2 + GE^2 = (EC^2 =)\ CD^2 + DE^2;$$
$$\text{and } DB^2 + DE^2 = (BE^2 =)\ BF^2 + FE^2; \text{whence,}$$

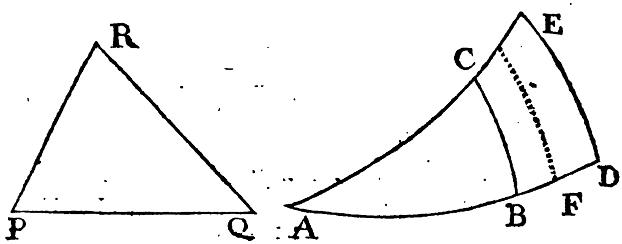


by

by adding each Side of the Equations together, we have $AF^2+GC^2+DB^2=AG^2+CD^2+BF^2$. Q. E. D.

PROBLEM LI. *Answered by* Mathematicus.

Let ABC and ADE be the two Triangles, F the Point of Bisection, and let PQR be a plain Triangle whose Angles P and Q are respectively measured or expounded by the Arches AB and AD.

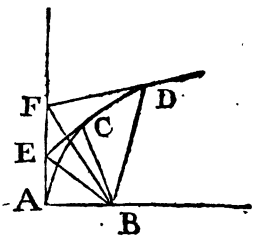


Then, it will be RQ : RP : : Sine of P (AB) : Sine of Q (AD) and Sine of AB : Sine of AD : : Tangent of BC : Tangent of DE *(by two well known Axioms)* : Whence, by Equality, RQ : RP : : Tangent BC : Tangent DE. Therefore, by Equality and Prob. 3. N°. 3. Tangent BC : Tangent DE : : Radius : Tangent of an Arch, and as Radius : Tangent of the Excess of this Arch above 45° : : Tangent $\frac{P+Q}{2}$ $(=\frac{AB+AD}{2}=AF)$: Tangent $\frac{Q-P}{2}$ $(=\frac{AD-AB}{2}=BF)$; which are the very Proportions that were to be demostrated.

PROBLEM LII. *Answered by* Mathematicus.

Let BC and BD be the given Lines, and let CE and DF be Tangents to the Curve meeting AF, the Tangent at the Vertex A, in E and F; also let BE and BF be drawn, which will respectively bisect

bisect the Angles ABC and ABD, by a known Property of the Parabola, and therefore EBF will also be equal to $\frac{1}{2}$CBD. Moreover *(by another known Property)* $\sqrt{BC} : \sqrt{BD}$ :: Radius : Tangent of an



Arch; and as Radius : Tangent of the Excess of this Arch above 45° :: Tangent $\frac{BEF+F}{2}$ ($=$ Tangent $\frac{180^\circ - EBF}{2} =$ * Tangent $90^\circ - \frac{CBD}{4} =$ Cotan. $\frac{CBD}{4}$) : Tangent of the required Arch $Q = \frac{BEF - F}{2} = \frac{90^\circ + ABE}{2} - \frac{90^\circ - ABF}{2} = \frac{ABE+ABF}{2}$, therefore $2Q = ABE + ABF$, and $2Q + EBF$ ($\frac{1}{2}$CBD) $= 2ABF = ABD$, and $2Q -$ EBF ($\frac{1}{2}$CBD) $=$ ABC. Q. E. D.

PROBLEM LIII. *Answered by* John Turner.

The Number of Chances for throwing 35, 36, 37 or 38 Points with ten Dice being 4395456,

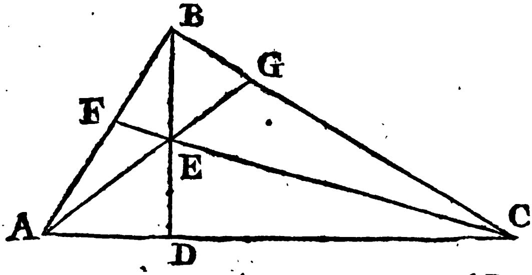
* For ABE+EBF=FBC+CBD; but ABE=EBF+FBC, therefore FBC+2EBF=FBC+CBD, consequently EBF$=\frac{CBD}{2}$.

4325310

4325310, 4121260, and 3801535 respectively (*as appears in Prob.* 22. *Simpson's Laws of Chance)* their Sum 16643561 divided by 60466176 will express the Probability of the proposed Event happening the first Throw, therefore the Probability of the contrary will be $\frac{43822615}{60466176} = 1 - \frac{16643561}{60466176}$; whence, the Probability of not happening in any Number (n) of Throws is $\overline{\frac{43822615}{60466176}}\Big|^{n}$, consequently the Probability of its happening in n Tryals will be $1 - \overline{\frac{43822615}{60466176}}\Big|^{n}$; whence the Odds as $\overline{\frac{43822615}{60466176}}\Big|^{n}$ to $1 - \overline{\frac{43822615}{60466176}}\Big|^{n}$; which, when $n = 3$, will be as $\overline{\frac{43822615}{60466176}}\Big|^{3}$ to $1 - \overline{\frac{43822615}{60466176}}\Big|^{3}$.

PROBLEM LIV. *Answered by* John Turner.

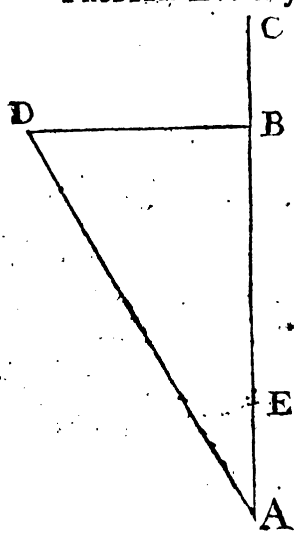
Let ABC be a Triangle similar to the proposed one, and let $AB = x$, $BF = BG = y$, and $BC = 1$; then will $AF = x - y$ and $CG = 1 - y$. Now by



similar Triangles $AC : AB :: AB : AD = \frac{AB^2}{AC}$, and $AC : BC :: BC : DC = \frac{BC^2}{AC}$. But (*by Theor.* 16. *Book*

Book 4. *Simpson's Geom.)* $DC \times AF \times BG = AD \times BF \times GC$; therefore because $BF = BG$, $DC \times AF$ will be $= AD \times GC$, and by substituting for AD and DC their Equals as above, we shall have $AF \times \frac{BC^2}{AC} = CG \times \frac{AB^2}{AC}$, *that is*, $1 \times \overline{x-y} = (x^2 \times \overline{1-y}) = x^2 - yx^2$; therefore $x - x^2 = y - yx^2$, whence $y = \frac{x-x^2}{1-x^2} = \frac{x}{1+x}$. Now $1 + \frac{x^2}{\overline{1+x}|^2} = CF^2$ and $x^2 + \frac{x^2}{\overline{1+x}|^2} = AG^2$, therefore by similar Figures $1 + \frac{x^2}{\overline{1+x}|^2} : x^2 + \frac{x^2}{\overline{1+x}|^2} :: a^2 : b^2$ (*a* and *b* representing the two given Lines in the required Triangle) consequently $a^2x^4 + 2a^2x^3 \begin{smallmatrix}+2a^2\\-2b^2\end{smallmatrix} x^2 - 2b^2x = b^2$; whence x and from thence the Sides of the Triangle required may be found.

PROBLEM LV. *Answered by* John Turner.

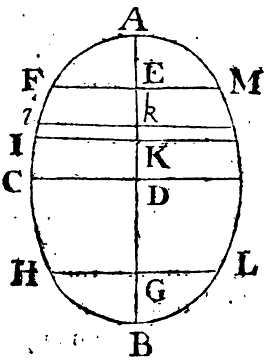


Suppose the Line CA to represent the River, C the Boat going down it and D the Boat that is to cross it; then, if DB be put $= a$, $BC = b$, the Velocity of D to that of C as m to n, and $AD = x$, we shall have $AB = \sqrt{x^2 - a^2}$ and $AC = b + \sqrt{x^2 - a^2}$. But $m : n :: x : \frac{nx}{m} = CE$ the Distance sail'd by C, therefore $b + \sqrt{x^2 - a^2} - \frac{nx}{m} = AE$ must be a Maximum

imum, and its Fluxion $\frac{x\dot{x}}{\sqrt{x^2-a^2}} - \frac{n\dot{x}}{m} = 0$; whence $\frac{x}{\sqrt{x^2-a^2}} = \frac{n}{m}$, which reduced gives $x = \frac{na}{\sqrt{n^2-m^2}}$.

PROBLEM LV. *Answered by* John Turner.

First, in order to find the Time in which any given spheroidal Cask FMLH will be emptied, let

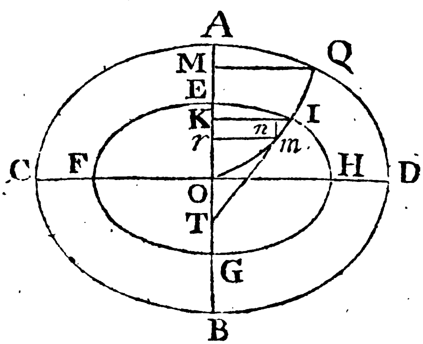


$AD=a$, $CD=b$, $DE=DG=c$, and $GK=y$: Then, if p be put $=.7854$, we shall have $IK^2 = \frac{b^2}{a^2} \times \overline{a^2 - \overline{y-c}|^2}$, and the Area of the Circle, described by KI, $= \frac{4pb^2}{a^2} \times \overline{a^2 - \overline{y-c}|^2}$, which multiplied by $\dot{y}$ ($=Kk$) gives $\frac{4pb^2\dot{y}}{a^2} \times \overline{a^2 - \overline{y-c}|^2}$ for the Fluxion of the Solidity; this divided by $\sqrt{y}$ gives $\frac{4pb^2}{a^2}$ $\times \overline{a^2 y^{-\frac{1}{2}}\dot{y} - y^{\frac{3}{2}}\dot{y} + 2cy^{\frac{1}{2}}\dot{y} - c^2 y^{-\frac{1}{2}}\dot{y}}$ for the Fluxion of the Time of Evacuation, whose Fluent, when y becomes

equal to $2c$, will be equal to $\frac{4pb^2}{a} \times \frac{30a^2 - 14c^2}{15} \times \sqrt{2c}$ the Time of Evacuation in the Cask whose Altitude is $2c$. Now, in the Case proposed, this Time must be a Maximum, therefore its Fluxion (supposing c variable) $\frac{30a^2\dot{c}}{\sqrt{2c}} - 28c\dot{c} \times \sqrt{2c} - \frac{14c^2\dot{c}}{\sqrt{2c}} = 0$; whence $30a^2 = 70c^2$, and $c = a\sqrt{\frac{3}{7}}$, which substituted in EF $(= \frac{b}{a}\sqrt{a^2 - c^2})$ gives $\frac{b}{a}\sqrt{a^2 - \frac{3a^2}{7}} = \frac{2b}{\sqrt{7}} = \frac{1}{2}$ the head Diameter.

PROBLEM LVII. *Answered by* John Turner.

Suppose ADBC to be the given Ellipsis, the Square of whose lesser Diameter is to that of its greater as 1 to m, and from any Point M, in the

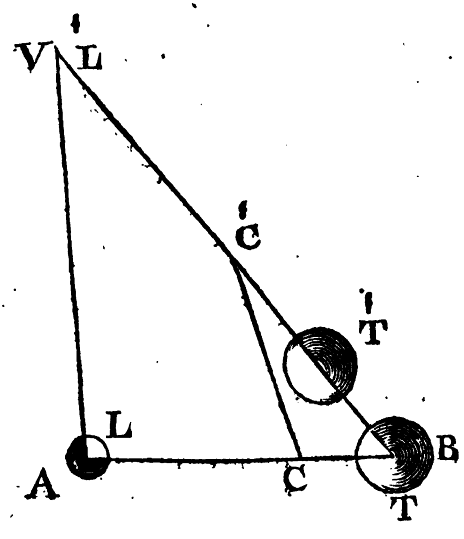


Semiconjugate OA, let MQ be drawn parallel to CD; then suppose FEHG to be an Ellipsis similar and concentric to the former, and draw KI parallel to CD, and mr infinitely near thereto; putting OM

OM$=b$, MQ$=c$, EO$=x$, KO$=z$, and KI$=y$, and we shall have KT, by the Property of the Ellipsis, $=mz$. But, by similar Triangles, $\dot{y} : \dot{z} :: y : mz$, therefore $mz\dot{y}=y\dot{z}$, $\frac{m\dot{y}}{y} = \frac{\dot{z}}{z}$, and m Log. $y=$ Log. $z : + D$. Now when z becomes equal to OM and y equal to MQ, then m Log. $c=$ Log. $b : + D$, whence $D = m$ Log. $c -$ Log. b; consequently m Log. $y=$ Log. $z + m$ Log. $c -$ Log. b, and $y^m = z \times \frac{c^m}{b}$, which shews that the Curve is a Parabola.

PROBLEM LVIII. *Answered by* Mathematicus.

In the annexed Scheme; let L and T represent the Bodies as projected, whose Masses are respec-



tively equal to those of the Moon and Earth, or in the Ratio of Unity to 39.778; and let AV and BV, the absolute Directions of the two Bodies meet each other in V: Then, in the Triangle ABV are given

 all

all the Angles and the Side AB (=100000 Miles); whence AV will be found equal to 253208.8163, and BV=287938.4609 Miles respectively.

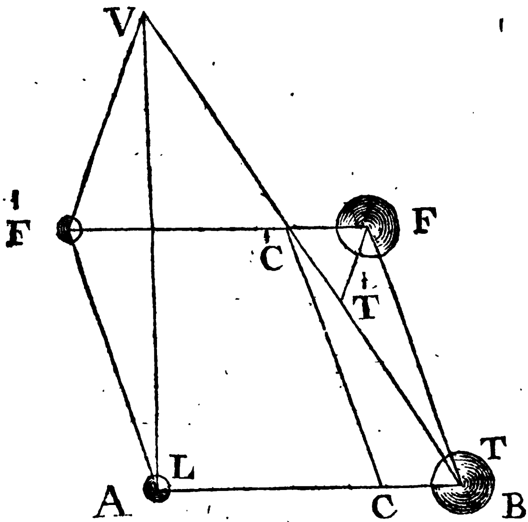
Now as 5 Miles, the absolute Space described by the Body L in one Second, is to 2 Miles, the Space described by the Body T in the same Time, so is the Measure of AV, in Miles, to 101283.5265, the Measure of the Space, in Miles, which T would uniformly describe in the Time that L uniformly describes the Space AV.

Let this Space be denoted by BT′, then it is manifest, when the Body L, by an uniform Motion hath described the Space AV, and is arrived at V, the Body T will have described the Space BT′ in the same Time, and arrived at T′; therefore if T′V (=BV—BT′ = 287938.4609 — 101283.5265 = 186654.9344) be divided at C′ in the given Ratio of their Masses inversely, or C′T′ be taken to T′V in the Ratio of Unity to 1 + 39.778, then C′ will be the common Center of Gravity (of the two Bodies) in this Position, and the Distance C′T′ will be found = 4577.3440; which added to BT′ gives BC′=105860.8705, whence VC′=18077.5904.

Moreover, let BC be taken to BA, in the Ratio of Unity to 1+39.778, and then C will be the Point of Equilibrio or Center of Gravity of the two Bodies L and T at the Instant of Projection, and consequently the Space CB will be found = 2452.3027 and AC = 97547.6973. Now, if the Right-line CC′ be drawn, it is plain, that in the Time the Bodies L and T would respectively describe the Space AV and BT′, their common Center of Gravity will describe the Space CC′; therefore in the Triangle BCC′, there being given the two Sides BC, BC′ and the included Angle CBC′=60°, the Angle BCC′, shewing the Direction of the Path of the Center of Gravity, will be found = 118° 52′, as also the Space CC′ (104656.2697 Miles) described by the same Center in the Time that L uniformly

uniformly describes the Space AV. Hence, it will be as AV : CC′ :: 5 Miles, the Space described by L in one Second, to 2.0666 Miles, the Space described by the Center of Gravity in the same Time.

Furthermore, draw BF equal and parallel to CC′, and let the Parallelogram be completed, and join V, F′ and F, F′; then in the Triangle F′C′V are



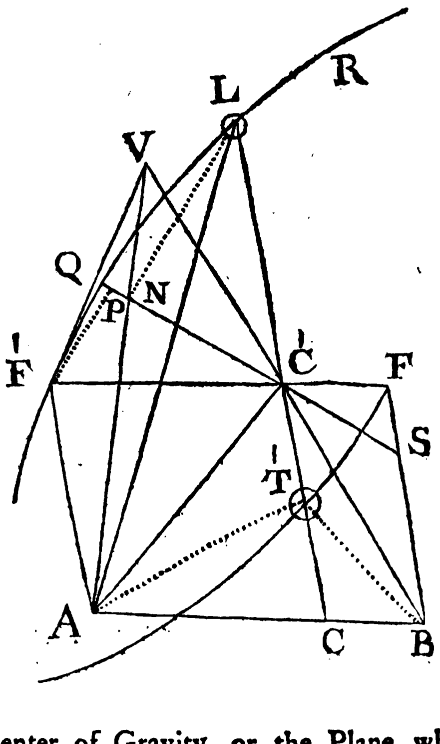
given the two Sides C′F′ (AC), C′V and the included Angle F′C′V (60°) whence the Angle C′F′V, the relative Direction of L with regard to the common Center of Gravity, will be found 86° 44′ 50″, and the Side F′V=157938.2468 Miles the relative Distance; hence it will be AV : VF′ :: 5 Miles, the absolute Velocity of the Body L in one Second : 3.118735 Miles, the relative Velocity in the same Time.

Moreover, since the Triangles VC′F′, and FC′T′ have one Angle C′ common, and the Sides about that Angle proportional, those Triangles are similar; therefore the Bodies L and T will describe similar Figures (which in this Case are Hyperbolas) about their common Center of Gravity; and the Space

C′F

C'F being to C'F' as 1 to 39.778; the relative Velocity of T will be to that of L in the same Ratio.

Let now F'LR represent the Trajectory, or Conic Section described by the Body L, to which F'V will be a Tangent at the Point F'; and let F'C'L be the Area described about C' the Focus, or Center of Force, in 48 Hours the proposed Time, and let CC' (=AF'=BF) be the Distance which the com-



mon Center of Gravity, or the Plane where the Motions are performed, is carried uniformly in that Time; also let Q be the Vertex of the Trajectory, and draw C'Q, to which draw F'P and LN perpendicular, putting the Sine of 86°. 44'. 50". the Angle C'F'V (to the Radius 1) (=.9983897=*s*;

3982 the Number of Miles in the Earth's Radius $=b$; 0.003046, the Parts of a Mile which a heavy Body will descend in a Second of Time at the Earth's Surface $=r$; $\sqrt{r}=0.0051913$; 97547.6973 (C'F') the Distance of the Point of Projection from the Center of Force $=d$; 3.118735, the relative Velocity of L in a Second, $=v$; the transverse Diameter of the described Section $=2a$; Conjugate $=2c$; Eccentricity $=e$; then (*by Pa.* 26. *Simpson's Essays.*) we have $2a=\dfrac{d}{\dfrac{dv^2}{4rb^2}-1}=24941.05$; $2c=$

$$\frac{\overline{2a}|^{\frac{1}{2}}\,vsd}{b\sqrt{r}}=218263.41;\ e=\sqrt{a^2+c^2}=109841.9;$$

whence the Distance of the Focus from the Vertex $=e-a=97371.38\ (=g)$.

Now to find the Area described about the Center of Force in 48 Hours, put $QN=x$, $LN=y$; the Names of the given Lines remaining as above: Then, because by the Property of the Curve

$$\frac{c^2\times\overline{2ax+x^2}}{a^2}=y^2,\ \text{we have}\ x=\frac{a\sqrt{c^2+y^2}}{c}-a;$$

therefore $\dot{x}=\dfrac{ay\dot{y}}{c\sqrt{c^2+y^2}}$, and $y\dot{x}=\dfrac{ay^2\dot{y}}{c\sqrt{c^2+y^2}}=\dfrac{a}{c}\times$

$$\frac{y^3\dot{y}}{\sqrt{c^2y^2+y^4}}=\frac{a}{c}\ \text{into}\ \frac{\frac{1}{2}c^2y\dot{y}+y^3\dot{y}}{\sqrt{c^2y^2+y^4}}-\frac{\frac{1}{2}c^2y\dot{y}}{\sqrt{c^2y^2+y^4}},$$ the

Fluxion of the Area QLN; whose Fluent $\dfrac{a}{c}$ into

$$\frac{\sqrt{c^2y^2+y^4}}{2}-\frac{c}{2}\times\text{Hyp. Log.}\ \frac{y+\sqrt{c^2+y^2}}{c}=$$ the

Area of QLN, to which adding $\dfrac{\overline{g-x}\times y}{2}$, the Area

of

of the Triangle CLN, we shall have $\frac{\overline{g+a}\times y}{2}-\frac{ac}{2}\times$ Hyp. Log. $\frac{y+\sqrt{c^2+y^2}}{c}$ for the Area of the hyperbolical Sector QCL. But, since F'C' is given $=d$, by the Property of the Curve, it will be as $e:a::$ $a+d-e:$ QP $::d-g:$ QP ($=40.0351$) because $a-e=-g$, as demonstrated in De L. *Hospital's* Conic Sections Prop. 1. Book 3; whence F'P will be found $=6493.1924$; which let be denoted by m, and substituted instead of y in the general Expression for the Area of the hyperbolical Sector, and there will come out $\frac{\overline{g+a}\times m}{2}-\frac{ac}{2}\times$ Hyp. Log. $\frac{m+\sqrt{c^2+m^2}}{c}$ for the Area of the Sector QC'F', which added to that of the former gives $\frac{\overline{g+a}\times\overline{m+y}}{2}-\frac{ac}{2}\times$ Hyp. Log. $\frac{\overline{m+\sqrt{c^2+m^2}}\times\overline{y+\sqrt{c^2+y^2}}}{c^2}$ the Area described about the Center of Force in 48 Hours, the given Time; which Area is likewise found by multiplying $\frac{vsd}{2}$, the Area described in one Second, by R the Number of Seconds in the given Time, wherefore $\frac{\overline{g+a}\times\overline{m+y}}{2}-\frac{ac}{2}\times$ Hyp. Log. $\frac{\overline{m+\sqrt{c^2+m^2}}\times\overline{y+\sqrt{c^2+y^2}}}{2}=\frac{vsRd}{2}$, from whence y will be found $=499651.2$ and consequently the Distance (LC') of the Body L from C' the Center of Force $=502296$. But LC' :

TC'

T'C.:: 39.778 : 1, therefore $T'C' = \frac{LC'}{39.778} =$ 12627, and consequently T'C'+LC', the absolute Distance of the Bodies T', L from each other = 514923.

Since in the two right-angled Triangles C'LN, C'F'P there are given two Sides in each of them, the Angles LC'N and NC'F' will be found = 84°. 7'. and 3°. 49'. respectively; the Sum of both which is 87°. 56'. the true Anomaly of each Body, or the Angle described about the Center C' in the given Time; but the Distance CC' uniformly described by the Center of Gravity of the two Bodies in that Time is 2.0666×48×60×60=357210.848 Miles; therefore if AC' and BT' be drawn, in the Triangle AC'F' will be given the two Sides AF' (=CC'), F'C' and the Angle AF'C' = (180° − 118°. 52') 61°. 8'; whence AC' = 103020, the Angle C'AF' = 56°. 2'. and, consequently, the Angle AC'F' = 62°. 50'; therefore in the Triangle AC'L are given the two Sides AC', C'L and the included Angle AC'L = 150°. 46', whence the Distance of the Body L from the Place of Projection A will be found = 594320 and the Angle C'AL = 24°. 23'. and consequently the Angle BAL, or the Position of the Body L = 87°. 13'.: Lastly, in the Triangle AC'T' are given the two Sides AC', C'T' and the included Angle AC'T', whence AT' = 97489 and the Angle T'AC' = 6°. 10'; therefore in the Triangle ABT' are given the two Sides AB, AT' and the contained Angle BAT', whence BT', the Distance of the Body T' from the Place of Projection B, will be found = 93785 and the Angle of Position ABT' = 60°. 17'. Q. E. I.

A COLLECTION OF PROBLEMS

To be anſwered in the next Number.

Problem LIX. *by* John Turner.

THREE Ships ſail from three different Ports, whoſe Latitudes are 14°, 12° and 10°. reſpectively; now ſuppoſing that, after the firſt Ship (which ſaild at the Rate of five Miles *per* Hour) had been thirteen Hours under ſail, the ſecond ſet out ſailing E. N. E. at the Rate of ſix Miles *per* Hour; alſo that, after the ſecond had been out two Hours, the third moved off with a Velocity of ſeven Miles *per* Hour, and after ſailing 23 Hours fell in with the other two at the ſame Inſtant of Time; 'tis required from thence to determine the Latitude of the Place arrived in, with the ſeveral Courſes and Departures of the firſt and third, likewiſe the Departure of the ſecond.

Problem

PROBLEM LX. *by* Jack Sinbad, *of St.* John's *Harbour in* Antigua.

Admit *Antigua* bears from *Montserrat* N. N. E, and that a Ship bound from the latter to the former sails, as near as she can lie to an East Trade-wind, with her starboard Tacks on board 'till she is to the Northward of *Antigua* 9.24 Miles: Now, supposing that her nearest Distance, upon that Tack, to *Antigua* is 3.87 Miles, and that the Sum of the Distance sailed from the Place where she is nearest to *Antigua*, and the Distance sailed with her larboard Tacks on board is 18 Miles; 'tis required to determine the Distance sailed on each Tack, and how near the Wind she has made her Course good.

PROBLEM LXI. *by* John Turner.

The Sum of the Squares of two Numbers being 41, and the Sum of their Cubes 189; what are the Numbers?

PROBLEM LXII. *by* John Turner.

To determine the Ratio of the Altitude, and base Diameter of the greatest Cylinder that can be cut out of a given Paraboloid of any Kind.

PROBLEM LXIII. *by* John Turner.

If the greatest horizontal Range of a Piece be 2000 Yards; at what Distance (supposing the Elevation and Charge of Powder to continue the same) will it be able to strike an Object elevated 54 Yards above the Plane of the Horizon?

PROBLEM LXIV. *by Mr.* Thomas Moss.

The Difference of the Segments of the Base, the Difference between the Perpendicular and the lesser Segment, and the Ratio of the Perpendicular and the greater Side of any plane Triangle being given; to determine the Triangle.

PROBLEM LXV. *by Mr.* William Kingston.

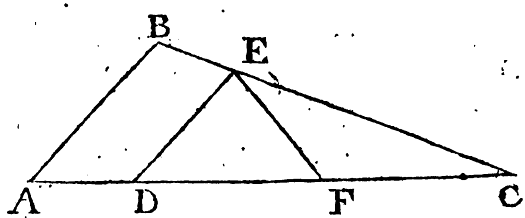
The Perpendicular of any plane Triangle, the vertical Angle, and the Angle formed by two Right-lines drawn from the Extremities of the Base to the Middle of the Perpendicular, being given; to determine the Triangle.

PROBLEM LXVI. *by* John Turner.

One Side of a Triangle, together with the Radii of its circumscribing and inscribed Circles, being given; to construct the Triangle geometrically.

PROBLEM LVII. *by* John Turner.

Suppose that, in the Triangle ABC, DE is parallel to AB and EF bisects DC; also that AB, BE, AD, EF and the Angle FEC are given; 'tis required to determine the Triangle.



PROBLEM LXVIII. *by Mr.* Thomas Moss.

Two Right-lines, meeting in a Point, being given both in Position and Length; to draw a Right-line thro' the Point of Concourse, so that if Perpendiculars be let fall thereon from the Ends of the two given Lines, the two Triangles formed thereby shall be equal.

PROBLEM

PROBLEM LXIX. *by Mr.* Thomas Moſs.

To find the Hour, Minute, Second and Third on *March* 10, when the Suns Altitude in the Latitude of *London* is a Maximum.

PROBLEM LXX. *by* John Turner.

Suppoſing the Latitudes of two Places, together with their Difference of Longitude, to be given; 'tis required to determine the Sun's Declination when he ſets to the Inhabitants of the two Places at the ſame Inſtant of Time.

PROBLEM LXXI. *by* B. Oxon.

In what Latitude will a Right-line, drawn from the Point of Suſpenſion of a Pendulum and continued through the Earth's Center, make the greateſt Angle poſſible with the Pendulum; and how great will this Angle be, ſuppoſing the Ratio of the equatoreal Diameter to the Polar to be as 230 to 229?

PROBLEM LXXII. *by Mr.* Thomas Moſs.

There are two Places, under the ſame Meridian at the Diſtance of 80 Miles from each other; where if two equal Staves $16\frac{1}{2}$ Feet in Length be erected perpendicular to the Horizon, it will ſo fall out, that on a certain Day of the Year, the Area of the Ellipſis deſcribed by the Shadow of the Northermoſt will be an Acre, and that deſcribed by the Shadow of the Southermoſt twice as great; 'tis propoſed from hence, to determine the Latitude of each Place, and alſo the Declination of the Sun when this happens.

Problem

PROBLEM LXXIII. *by* T.

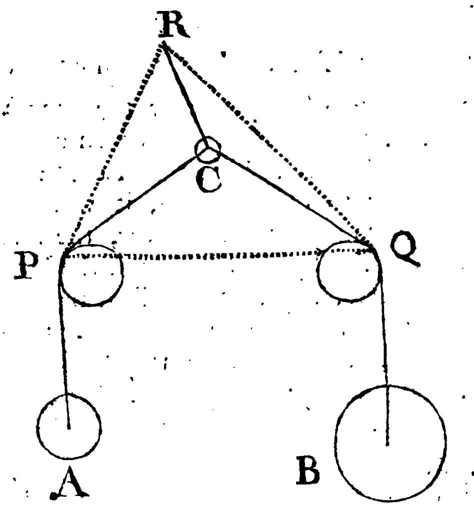
To determine at what Time of the Year the meridional Shadows of Objects, from Noon to Noon, admit of the greatest Increase; the Latitude of the Place being given (=51°. 32′.) and the Motion of the Sun in the Ecliptic considered as equable.

PROBLEM LXXIV. *by* John Turner.

There is a certain Place on the Surface of the Earth, from whence a heavy Body, descending in a Right-line, will fall to the Center in one Second of Time less, than if it had descended from a Place under the Equator; 'tis required from thence to determine the Latitude of the Place, supposing the earth an oblate Spheroid whose equatoreal Diameter is 7974 and polar Diameter 7940 Miles.

PROBLEM LXXV. *by* John Turner.

Suppose a given Weight C to be suspended by Means of a String to a given Point R, and supposing two



other

other given Weights A and B to act upon the former by two Strings paſſing over two Pullys at the given Points P, Q in the ſame horizontal Line PQ; 'tis required to determine the Poſition of each Weight when they are in Equilibrio.

Problem LXXVI. *by* T.

To determine the Gravitation at any Point in the produced Axis of a given Solid, the Attraction of each Particle of Matter in that Solid being as any Power *(n)* of the Diſtance.

The End of Number IV.

THE

Mathematician.

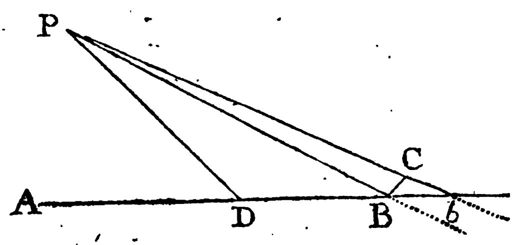
DISSERTATION V.

Upon the Progreſs and Improvement of GEOMETRY.

IN our laſt we attempted to give an Account of that ~~great modern~~ Improvement of Sir *Iſaac Newton*, called the Doctrine of Fluxions; in proſecuting whereof, we firſt explained the Nature, and gave a Definition of it, and ſecondly ſhewed, in ſome Inſtances the Manner of determining what thoſe finite Magnitudes are, by which the Relations of Fluxions may be expreſſed; which finite Magnitudes (in *uniform* Motions) appeared to be *the Augments of their Fluents generated in equal Particles of Time*; and (in

accelerated or retarded Motions) after explaining the Nature of prime and ultimate Ratios, we shewed that the Magnitudes required, where those that are *in the first Ratio of the nascent Augments, or the last Ratio of the evanescent Parts.* We shall now exhibit some more Instances of determining such Magnitudes as are proportional to, and expressive of, Fluxions, generated by *accelerated* or *retarded* Motions, by the help of the Doctrine of prime and ultimate Ratios before explained, and then proceed to the third Division proposed to be treated of in our last Dissertation.

The Examples we shall now give, are those Mr. *Ditton* has used, in his Institution, where he has happily elucidated the fundamental Principles, and Algorithm or Manner of operating in this Method; pity it is that he droped his Readers when he came to the Use and Application of Fluxions; for there is no doubt that his able Hand could have treated this latter Part, in as familiar, explicit, and perspicuous a Manner as he has done the Principles.

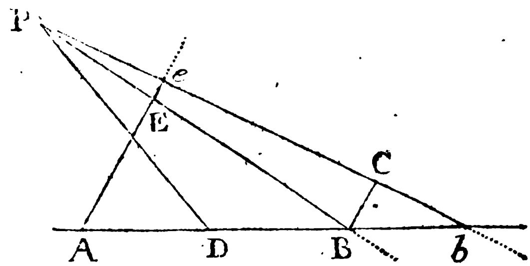


Let the right Line PB turn round upon the Poin P as a Center, and continually cut another righ Line as AB given in Position; 'tis required to find the Proportion of the Fluxions of these right Lines AB and PB. Suppose the Line PB to move out of its Place by its revolving Motion, and to come into a new Place P*b*. In P*b* let us take PC=PB, and joining the Points B, C, let the Line PD be drawn,

drawn, so as to make the Angle bPD=the Angle bBC; then will the Triangles bDP and bCB be similar. Now Cb is the Increment of the Line PB, as Bb is that of the Line AB, and both these Increments are evidently generated in the same Particle of Time; for while the Line PB by flowing is augmented into Pb, the Line AB is also augmented into Ab. From the similar Triangles it is Bb : Cb :: Pb : Db; that is, (since Pb=PC+Cb, and Db= DB+Bb) Bb : Cb :: PC+Cb : DB+Bb; this is the Proportion of the finite Augments. Now to obtain the last Ratio of these Augments (considered as *vanishing*, or which is all one, the first Ratio of them considered as *arising*,) we must imagine the Line Pb to return back into its former Place PB, by which Means the Augments Bb and Cb will vanish, and become equal to nothing. Then if in finite Terms the Ratio of Bb to Cb, be equal to the Ratio of PC+Cb to DB+Bb, certainly the ultimate Ratio of Bb to Cb, just in that particular Case when vanishing, can be no other than the Ratio of PC to DB, or its equal PB to DB (PB being=PC,) because when the Augments are vanished, the Expression of the Ratio comes to this. But Fluxions are in the last Ratio of the evanescent Augments, and consequently the Ratio of the Fluxions of AB and PB, is equal to the Ratio of PB to DB, that is, they are one to another as PB to DB.

Again, Let the right Line PB turning about the Point P as a Centre, intersect the two right Lines AB and AE (given in Position) in the Points E, and B: 'Tis required to determine what Proportion the Fluxions of those Lines AB and AE have, when the Lines themselves are generated by such a Motion. Suppose, as before, the Line PB to move from the Place PB into the new Place Pb, which cuts the Lines AB and AE in the Points b, &c. Draw BC parallel to AE intersecting Pb in the Point C. Then

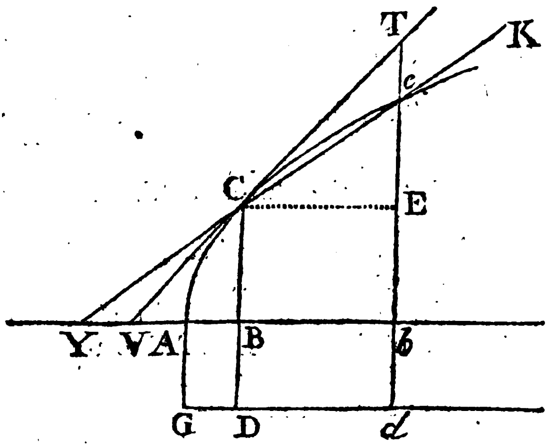
are the Triangles BbC, and Abe similar, as also the Triangles PBC, and PEe. 'Tis plain that Bb and Ee are the Augments of the Lines AB and AE generated in the same Time. Now from the similar



Triangles BbC, Abe, 'tis $Bb : BC :: Ab : Ae$, whence $Bb = \frac{BC \times Ab}{Ae}$; and from the similar Triangles PBC, PEe, it is, $PB : PE :: BC : Ee$; whence $Ee = \frac{PE \times BC}{PB}$. Consequently $Bb : Ee :: \frac{BC \times Ab}{Ae} : \frac{BC \times PE}{PB}$ $:: \frac{Ab}{Ae} : \frac{PE}{PB} :: \frac{AB+Bb}{AE+Ee} : \frac{PE}{PB}$ (for $Ab = AB+Bb$, and $Ae = AE+Ee$.) Wherefore the Ratio of the finite Augments, *viz.* $\frac{Bb}{Ee}$ being $= \frac{\overline{AB \times PB} + \overline{Bb \times PB}}{\overline{AE \times PE} + \overline{Ee \times PE}}$; therefore the ultimate Ratio of $\frac{Bb}{Ee}$ is $= \frac{AB \times PB}{AE \times PE}$; for now the Augments Bb and Ee are supposed to vanish, and consequently the Terms drawn into them, vanish also. Therefore the Proportion of the Fluxions, is the same with that of these Rectangles; or the Fluxion of AB, to the Fluxion of AE is as $AB \times PB : AE \times PE$.

Again,

Again, to shew how the Proportion and Expression of Fluxions in curvilinear Figures are to be derived and demonstrated, from this first Principle of prime and ultimate Ratios.



Let ACc be any Curve whose Abscis is AB, Ordinate at right Angles CB, and Tangent at C, TCV; 'tis required to assign the Relation of the Fluxion of the Ordinate, to the Fluxion of the Abscis.

Suppose the Ordinate BC to move uniformly, and come into the new Place *bc*, and drawing CE parallel to AB, 'tis plain that the little Lines CE and *c*E are the Increments of the Abscis and Ordinate generated in the same Particle of Time; for while AB by flowing becomes A*b*, CB flows into *cb*. Draw the right Line *c*C subtending the curvilinear Arch *c*C, which Line *c*C produced till it cuts AB produced in Y: Then the rectilinear Triangles *c*CE, *c*Y*b* are similar; therefore CE : *c*E :: Y*b* : *cb* that is, CE : *c*E : : YB+B*b* : (or YB+CE; for CE=B*b*) CB+*c*E, wherefore in finite Terms the Ratio of the Augments, *viz.* $\frac{CE}{cE} = \frac{YB+CE}{CB+cE}$; suppose now the

Ordinate

Ordinate cb to return back into its first Place CB, or (which is all one) imagine the Points c and C to come together, then will the right Line cCY be coincident with the Tangent TCV, and so YB will become VB, and the rectilinear Triangle cCE, in its *last evanescent Form*, will be similar to the Triangle CVB. Therefore the ultimate Ratio $\frac{CE}{cE}$ will be $= \frac{VB}{CB}$, that is, the Fluxion of the Absciss, will be to the Fluxion of the Ordinate, as the Subtangent VB is to the Ordinate CB.

2dly. Let it be proposed to find the Proportion of the Fluxion of the curve Line AC to the Fluxion of the Ordinate BC. The Augments of these flowing Quantities generated in the same Time, are the little curvilinear Arch Cc, and the little right Line Ec. Now arguing with the right Line Cc (the Subtense of that little Portion of the Curve Cc) from the Similarity of the forementioned Triangles, we have $Cc : cE :: cY : cb$ or $Cc : cE :: cC+CY : cE+CB$, so that in finite Terms $\frac{Cc}{cE} = \frac{cC+CY}{cE+CB}$. But when the Points c and C come together, then the Secant CY will coincide with the Tangent CV; and the *evanescent* Triangle CEc will, in its *last Form*, be similar to the Triangle CVB; and the Sides of the one be proportional to the Sides of the other. Therefore the ultimate Ratio $\frac{\text{Subtense } Cc}{\text{right Line } cE}$ will be $= \frac{CV}{CB}$. But the ultimate Ratio of the Subtense Cc, to the Curve Cc, is a Ratio of Equality; so that the one may be taken for the other; there-

fore

fore the ultimate Ratio $\frac{\text{Curve } Cc}{\text{right Line } cE}$ will be $= \frac{CV}{CB}$;

that is, the Fluxion of the Curve, is to the Fluxion of the Ordinate, as the Tangent CV, to the Ordinate CB. By the very ſame way of reaſoning it will be found, that the Fluxion of the Curve, is to the Fluxion of the Abſciſs; as the Tangent, is to the Subtangent; the Fluxions of the Curve, Ordinate, and Abſciſs, being as the Tangent, Ordinate and Subtangent.

3dly. Let it be propoſed to find the Proportion of the Fluxion of the curvilinear Area ABC, to the Fluxion of the Rectangle ABGD. The Augments here generated in the ſame Particle of Time, are the little *curvilinear Trapezium* BCcb, and the *little Parallelogram* BDdb; and for Brevities Sake we'll call theſe Augments reſpectively A and a. Call the curvilinear Space included between the Curve Cc, and the Line CE, q; then is the curvilinear Trapezium $BCcb = BC \times Bb + q$; that is, equal to the Rectangle BCEb + the curve Space included between the Arch Cc, and right Line CE. So that $A : a :: BC \times Bb + q : BD \times Bb$ in finite Terms; that is as $BC + \frac{q}{Bb} : BD$. But when the Points C and c coincide, then the Space q vaniſhes. Therefore the ultimate Ratio $\frac{A}{a} = \frac{BC}{BD}$; for

$\frac{q}{Bb}$ goes out and vaniſhes entirely, and conſequently the Fluxion of the Area ABC, is to the Fluxion of the Area ABGD, as the Ordinate BC, is to the Ordinate BD.

Tho' we are ſenſible this Truth was proved in our laſt Diſſertation, in a very ſhort Manner, from the ſimultaneous Increments, we hope this laſt Method will not be thought ſuperfluous, eſpecially as

Specu-

Speculations purely geometrical, may perhaps be more intelligible and eafy, than thofe that are mixed with Metaphyfics, or Algebra.

4thly. Again, let us determine the Proportion of the Fluxion of the *Solid* generated by the Rotation of the curvilinear Area ABC about the Axe AB, to the Fluxion of the Solid generated by the Rotation of the Rectangle ABDG about the fame Axe. The Augments of thefe flowing Quantities generated in the fame Particle of Time, are the Solids generated by the Rotation of the little curvilinear Area B*Ccb*, and the little Rectangle BD*db*. But the Solid generated by the Area BC*cb* is to be conceived confifting of two others, even as the generating plain Figure confifts of two Parts, the Rectangle BCE*b*, and the curvilinear Triangle C*c*E. So the whole Solid produced by the Rotation, confifts of the Solid generated by the Rectangle BCE*b*, which is a Cylinder whofe Bafe is BC, the Altitude CE or B*b*; and the Solid generated by the curvilinear Triangle CE*c* which is a Sort of Ring or Annulus. Ufing the Symbols A, *a*, as before; let *q* now denote the little Solid generated by the Rotation of the curvilinear Space C*c*E about the Axe AB. Then fhall A and *a* be expreffed by $BC^2 \times Bb + q$ and $BD^2 \times Bb$; for the Circle defcribed by BC and BD are as the Squares of thofe Lines refpectively, and fo $BC^2 \times Bb$ is to the Cylinder defcribed by the Rectangle BCE*b* as $BD^2 \times Bb$ to the Cylinder defcribed by the Rectangle BD*db*. Since therefore $A : a :: BC^2 \times Bb + q : BD^2 \times Bb$, in finite Terms; that is, as $BC^2 + \frac{q}{Bb} : BD^2$; then fhall the ultimate Ratio $\frac{A}{a} = \frac{BC^2}{BD^2}$. For when the Points C and *c* coincide, the little Solid *q* defcribed by the curvilinear Triangle C*c*E, vanifhes. Therefore the Fluxion of the Solid defcribed by ABC, is to the Fluxion of the Solid defcribed by ABGD, as the Square of BC, to the Square of BD.

By

By the like Way of Reaſoning may be found, the Proportion of the Fluxions of the curve Surface, generated by the Rotation of the curve Line AC, to the Fluxion of the cylindric Surface, deſcribed by the right Line GD, revolving about the ſame Axis with the former, *viz.* AB; by reducing it from a curve Surface to a curvilinear Area: But we ſhall not trouble the Reader with any more Inſtances of this Kind, and hope the Importance of them will excuſe us from dwelling ſo long upon theſe Propoſitions, which to ſome may ſeem very plain, but are not therefore to be deſpiſed; for next to a juſt and clear Idea of Fluxions, there is nothing more neceſſary for completely apprehending this Doctrine, than the determining their Proportions; becauſe it will appear by and by, that from hence may be virtually deduced, one of the principal Operations in this Method, *viz.* the deriving a fluxionary Equation from a fluential one; tho' the ſame is generally done by certain practical Rules laid down in the Algorithm of Fluxions for Eaſe and Expedition.

If it ſhould be objected, that there can be no ultimate Ratio of continually, diminiſhing and at laſt evaneſcent Quantities; becauſe, before they vaniſh it is not the laſt; and after they vaniſh, they have no Ratio: The Anſwer, according to the great Inventor who foreſaw it, is this; that the ultimate Ratio is neither the Ratio of them, before they vaniſh, nor after they vaniſh; but the Ratio wherewith they vaniſh, or the Limit to which their varying Ratio no ſooner arrives, than they vaniſh. If there was any thing in this Objection, it would infer, that when a falling Body is ſtopt in its Motion, it has no laſt or ultimate Velocity; for the Velocity before it was ſtopt, is not the laſt, and the Velocity after it is ſtopt, is none at all. But every one may ſee, that by the laſt Velocity is meant, neither the one or the other of theſe; but the Ve-

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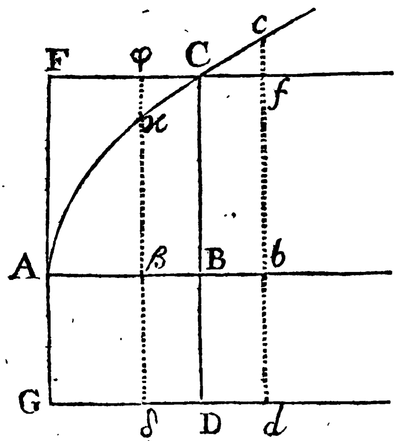
locity it has at that very Inſtant it ſtops, which it does not arrive at before it is ſtopt, and no ſooner arrives at but it is ſtopt. For ſince it moves with a continually accelerated Velocity, it has a different Velocity, for every different Inſtant of Time, and therefore at that Inſtant it ſtops, it has acquired a Velocity different from the Velocity it had at any other Inſtant of Time. And the ſame is the Caſe of the ultimate Ratio of evaneſcent Quantities, whoſe Ratio is continually varying; it is that Ratio they have at that very Inſtant they vaniſh: For, ſince they are ſuppoſed to have a different Ratio for every different Inſtant of Time, they muſt have a certain determinate Ratio, with which they vaniſh, otherwiſe they never can vaniſh, which is contrary to the Hypotheſis; and that is the ultimate Ratio of the evaneſcent Quantities.

It ſignifies nothing to ſay that ultimate Quantities cannot be aſſigned, in regard Quantity is diviſible without End; for it is not the Quantities themſelves that is hereby determined, but only their Ratio; which is capable of being determined.

Altho' this Method of inveſtigating and demonſtrating the Proportions of the Fluxions of plain Figures, is both certain and geometrical, yet ſuch perhaps may be the Scruples of ſome, and ſuch the Obſtinacy of others, againſt the whole of this Doctrine of prime and ultimate Ratios, of naſcent and evenaſcent Quantities; that we hope it will not be unacceptable to the Reader, if we ſhew, that the Proportion of the Fluxions of plain Figures, (and conſequently by Analogy of all their flowing Quantities) may be demonſtrated in another Manner, upon the moſt indiſputable Principles of Geometry, without introducing either infinitely little Quantities, as has been done without ſufficient Caution by ſome, but purpoſely avoided by Sir *Iſaac Newton*; or yet naſcent and evaneſcent Quantities, with their prime and

and ultimate Ratios, upon which Foundation he has built this Doctrine.

Let ABC and ABDG be a curvilinear Area and Parallelogram described by the uniform Motion of AG, and Ordinate CB, along the Abscis; and let ABCF be another Parallelogram described at the same time by a given Ordinate AF, so that all the three Areas ABC, ABDG, and ABCF be described by the uniform Motion of their respective Ordinates, lying always in the same right Line; then will *F*.ABC : *F*.ABDG :: BC : BD.



For since the Ordinates or Sides of the Parallelograms ABDG, ABCF are given or invariable right Lines, and these Lines are supposed to move uniformly along the common Abscis AB, the Fluxions or Velocities of flowing must be constant and invariable; for the Spaces generated in equal Times are evidently equal in the respective Parallelograms; and the Fluxions or Velocities of flowing in that Case are as the Increments generated in any the same Times, or by Composition as the whole Areas generated, that is, *F*.ABCF : *F*.ABDG :: (ABCF : ABDG)

ABDG) : : BC : BD. But the Velocity with which the curvilinear ABC is described, must be less than the Velocity wherewith the Parallelogram ABCF is described, at any Time before the Ordinate BC arrives to its present Position, as when it is in $\beta\kappa\phi$, because $\beta\kappa$ is less than $\beta\phi$: Again, the Velocity with which ABC flows, at any Time after ABC has past its present Position, as when it is in *bfc*, is greater than the Velocity with which ABC flows, since *bc* is greater than *bf*; and these Things are true, let $\beta\kappa\phi$ and *bfc* be as near to BC as you please, because the Curve AC and right Line FC intersect in one Point only; but the Velocity with which the Area ABC flows, is continually or incessantly varying, since the Ordinate BC never continues of the same Length, for any the least Time: Wherefore betwixt the two Instants of Time that the Ordinate BC has the Position $\beta\kappa$ and *bc*, the variable Velocity with which the Area ABC flows, must be in all the intermediate Degrees possible; therefore also in that very Degree of Velocity with which ABCF flows, which is invariable and intermediate: But it has been proved, that this cannot happen at any Time, when the Ordinate BC is out of the present Position, therefore it must happen when the Ordinate BC obtains the present Position. But the Fluxions of flowing Quantities are the Velocities with which they flow or increase; therefore, when the Ordinate BC comes to the present Position, the Fluxions of the curvilinear and rectilinear Areas ABC, and ABCF, are equal; but, as was before shewn, *F*. ABCF : *F*.ABDG : : BC : BD, therefore *F*.ABC : *F*.ABDG : : Ordinate BC : BD.

The Reasoning in this last Demonstration of Mr. *Stewart*'s, is analogous to that of Mr. *Simpson*'s, quoted in our last Dissertation, where the same Truth is proved; those that easily apprehend the general Reasonings of Algebraists about abstract Quanti-

Quantity, may perhaps prefer the former, but they that are most familiar with Speculations of Figures purely geometrical, will be pleased with the latter, especially as it is independent of infinite Series; and as the same Truth when set in different Lights is more likely to convince, than when in one View only, we hope to be excused from having dwelt so long upon the geometrical Proportions of Fluxions only, tho' it be a fundamental Consideration; we shall afterwards proceed to enquire what it will avail us to have those Proportions expressed in finite Terms. And here let us remember that Sir *Isaac* tells us, his principal Enquiry was, *to determine the Quantities themselves from the Relation of the Velocities with which they are generated*; we now see at large how he obtained this Relation or Proportion geometrically; but if we observe what he did with them when obtained, we shall find his Invention as prolific in Algebra, as it was penetrating in Geometry; never did his invincible Genius exert more amazing Efforts, than in completing this Method of Fluxions, by an Analysis that surmounted all Difficulties; it was then he enriched Algebra with all the various Management and Use of the Doctrine of infinite Series, and Discovery of his binomial Theorem, (now carved on his Monument in *Westminster Abby*, to perpetuate the Memory of his divine Genius) and pursued algebraic Quantity thro' all its serpentine Shapes and mazy Meanders, *sua face monstrante viam*.

Tho' the Idea of a Fluxion, as hitherto defined, seems more immediately applicable to geometrical Magnitudes, (which are naturally conceived to be generated by Motion,) than to Quantities considered abstractedly, or as they are expressed by general Symbols in Algebra; the Evidence of the Method also has been disputed, and Objections made to the Number of Symbols employed in it, as if they might

serve

ſerve to cover Defects in the Principles and Demonſtrations: Yet it is an important Part of this Doctrine of Fluxions, and as the Improvements that have been made by it, either in Geometry or Philoſophy are in a great Meaſure owing to the Facility, Conciſeneſs and great Extent of the Method of Computation or algebraic Part; it may therefore be proper for us to ſhew in this Place, in order to make a Tranſition from Geometry to Algebra, that all the Truths that have been proved in this Method to reſult from the Contemplation of geometrical Magnitudes, may with equal Force and Certainty be demonſtrated of abſtract Quantities in general, to ſhew the Univerſality of this Doctrine. Algebra is only a Kind of univerſal Arithmetic, and infinite Series are analogous to decimal Fractions; this Univerſality can never be ſuppoſed to derogate from its Evidence, if we have no Ideas more clear or diſtinct than thoſe of Numbers, and often acquire more ſatisfactory and diſtinct Knowledge from Computations than from Conſtructions; for without doubt, Obſcurity may be avoided in this Art, as well as in Geometry, by defining clearly the Import and Uſe of the Symbols, and proceeding with Care afterwards.

Any Quantities that are produced from each other by an algebraic Operation, or whoſe Relation is expreſſed by any algebraic Equation, being ſuppoſed to increaſe or decreaſe together, ſome will be found to increaſe or decreaſe by greater Differences, or at a greater Rate, others by leſſer Differences, or at a leſſer Rate; and while ſome are ſuppoſed to increaſe or decreaſe at one conſtant Rate, by equal ſucceſſive Differences, others increaſe or decreaſe by Differences that are always varying. Thus when a Quantity A increaſes by Differences equal to a, $2A$ increaſes or decreaſes by Differences equal to $2a$, then it manifeſtly increaſes or decreaſes at a greater Rate than A, in the Ratio of $2a$ to a, or of 2 to 1.

To

To accommodate the Doctrine of Fluxions to Quantity in general, we ſhall therefore now underſtand by the Fluxions of Quantities, *any Meaſures of their reſpective Rates of Increaſe or Decreaſe, while they vary or flow together*. There can be no Difficulty in determining thoſe Meaſures, when the Quantities increaſe or decreaſe by ſucceſſive Differences, that are always in the ſame invariable Proportion to each other; while A becomes A+*a* or A—*a*, 2A becomes 2A+2*a* or 2A—2*a*; and as 2A varies at a greater Rate than A, in the Proportion of 2*a* to *a*, ſo the Fluxion of A being expreſſed by *a*, the Fluxion of 2A muſt be expounded by 2*a*, or by any other Quantities proportional to them, as $\frac{1}{2}a$ and *a*.

When the Quantities increaſe or decreaſe by Differences that are not in the ſame Proportion to each other, theſe Meaſures may be commodiouſly determined by Sir *Iſaac*'s Moments; which in his *Method of Fluxions and infinite Series*, he defines thus: *The cotemporary Moments of flowing Quantities, are as the Velocities of flowing or increaſing, that is, as their Fluxions.* Now if this be proved true of Lines, it will equally obtain in all flowing Quantities whatever, which may be always adequately repreſented and expounded by Lines. But in equable Motions, the Times being equal, the Spaces deſcribed will be as the Velocities of Deſcription, as is known in Mechanics: And if this be true of any finite Spaces whatſoever, or of all Spaces in general, it muſt alſo obtain in indefinitely little Spaces, which we call Moments. And even in Motions continually accelerated or retarded, the Motions in indefinitely little Spaces, or Moments, muſt degenerate into Equability: So the Velocities (or Meaſures of their reſpective Rates) of Increaſe and Decreaſe, or the Fluxions, will be always as the cotemporary Mo-

ments. Therefore the Ratio of the Fluxions of Quantities, and the Ratio of their cotemporary Moments, will always be the ſame, and may be uſed promiſcuouſly for each other.

For the commodious Expreſſion of the Fluxions of Quantities in general, eſpecially thoſe whoſe Fluents vary by Differences continually varying, Sir *Iſaac* inſtituted an Analyſis for operating in Fluxions, manageable by, and conſiſtent with the Rules of common Algebra. We ſhall here give ſome Account of their Notation, becauſe it was our profeſs'd Deſign, to lay down the Rationale or Theory; but for the practical Rules ſhall refer our Readers to thoſe Authors who have taught the Algorithm thereof, *viz. Hayes*, *Ditton*, *l'Hoſpital*, *Simpſon*, *Muller*, *Stone*, *Emerſon*, &c.

The conſtant or determinate Quantities, are denoted by a, b, c, d, &c. the firſt Letters of the Alphabet; flowing or variable Quantities or Fluents, by v, x, y, z, &c. the laſt Letters thereof; and their Fluxions, or Celerities or Rates of Increaſe or Decreaſe, by the ſame Letters with a Dot over their Heads, as $\dot{v}$, $\dot{x}$, $\dot{y}$, $\dot{z}$, &c.

The ſynchronal Augments, or momentaneous Increments of v, x, y, z, generated in the firſt or laſt equal Tempuſculum of their Exiſtence, are repreſented by the Expreſſions $o\dot{v}$, $o\dot{x}$, $o\dot{y}$, $o\dot{z}$, and are very properly and ſignificantly called *Moments*. This Term *Moment*, we know in its common Uſe, intimates a product of Magnitude into Velocity, and appears to do the ſame here. For $o\dot{z}$ is the Product of o multiplied into $\dot{z}$, that is, of the indefinitely ſmall Quantity o, multiplied into the Fluxion or Velocity of the *ariſing* Increment of the flowing Quantity z, and ſo of the reſt. And ſurely if there can be no Increment, how ſmall ſoever aſſigned, but there are yet Increments leſs and leſs, aſſign as ſmall ones as you pleaſe; (which is a natural Conſe-

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quence from the infinite Divisibility of Matter, mentally at least, tho' not actually;) and since we may form a Notion, not indeed of absolute, but of relative and comparative Infinity, we may certainly be allowed to assume a Notation denoting nascent or evanescent Increments, which are indefinitely small, altho' we have no positive adequate Idea of such Increments or Moments. For a negative Idea is sufficient in many Cases for determining the Properties and Proportions of Quantities: What other than a negative Idea have we of the infinite Decimal 6.666, *&c.*? Yet we can demonstrate that it is exactly $=\frac{2}{3}$. After the same Manner, tho' the momentaneous Increments of flowing Quantities, cannot be distinctly and adequately conceived by the Mind, yet we may make use of some Kind of Notation, by way of Symbol or Representation of them; and their Relations may be determined and expressed by finite Quantities, which are distinctly conceived.

Altho' the Increment of any one flowing Quantity may congruously enough be represented by one indefinite Quantity o; yet the Increments of several together cannot possibly be expressed by one and the same Quantity, because the *Law and Reason of flowing*, is not the same in all of them, but *different and various:* Therefore we cannot put $z+o$, $y+o$, $x+o$ for the Increments of z, y, x generated in a given Moment of Time: On the other Hand, neither can we represent the Increments of these flowing Quantities thus, *viz.* $z+\dot{z}$, $y+\dot{y}$, $x+\dot{x}$: for $\dot{z}$, $\dot{y}$, $\dot{x}$ denote the Fluxions, that is, the Velocities of the *arising* Increments of these flowing Quantities; and it would be disproportional to represent the Increments of any Quantities, by Symbols that denote mere Velocity: So that neither of these Ways of Expression can do alone and of themselves; but we may fairly represent these Increments by the Pro-

ducts $o\dot{z}$, $o\dot{y}$, $o\dot{x}$: For here is *Difference* and *Variety* in this Notation, that serves to express the different Increments of different, and differently flowing Quantities; because tho' one Factor o, is the same in all, yet the other Factors are all different, and consequently the whole *Moments* are so too. But then (which is the main Point) these *Moments* are in Proportion to one another, as the Fluxions of the flowing Quantities respectively, for $o\dot{z}$, $o\dot{y}$, $o\dot{x}$, are as $\dot{z}$, $\dot{y}$, $\dot{x}$; and Sir *Isaac* has expresly told us, that the Increments generated in a very small Particle of Time were *quam proxime*, *very nearly*, as the Fluxions. So that in Congruity to that fundamental Law, the Increments ought to be expressed so, as that their Expressions might be proportioned to those of the Fluxions; which could not be, but by representing them in this Manner, as *Moments* or Products. From whence the considerate Reader may see what wonderful Art and Contrivance there is, even in the most minute Steps taken by the great Author in this Method.

Tho' it seems evident to us, that the Notation and Use of *Moments* are sufficiently justified, yet as some Objections have been advanced against them, and particularly against Sir *Isaac*'s Manner of operating by them, in finding the Fluxion of a Rectangle; and as an adequate Conception of them is not easy; it may perhaps be acceptable to some, to have the following short historical Account of them, taken from Mr. *Robins*; and as it is not foreign to the Design of our Dissertation, we hope it will not be deemed a faulty Digression, tho' it does a little interrupt the Series of our Explication. About the Year 1666, Sir *Isaac* drew up a short Discourse *de Analysi per æquationes numero terminorum infinitas*; here the Word *Moment* frequently occurs. * He has

* *Phil. Trans.* No. 342.

told

told us, *this Tract teaches how to resolve finite Equations into infinite Ones, and how by the Method of Moments to apply Equations both finite and infinite to the Solution of Problems.* He ſays, *that he there called the Moment of a Line a Point, in the Senſe of Cavalerius, and the Moment of an Area, a Line in the ſame Senſe.* The Paſſage in the Book to which this relates, is as follows, *Nec vereor loqui de unitate in punctis, ſive lineis infinité parvis, ſiquidem proportiones ibi jam contemplantur Geometræ, dum utuntur methodis indiviſibilium*; that is, Nor am I afraid to ſpeak of Unity in Points or Lines infinitely ſmall, ſince Geometers do now conſider Proportions even in ſuch a Caſe, while they uſe the Methods of Indiviſibles. He had juſt been expounding theſe *Moments* by Unity. He has alſo told us, *from the Moments of Time, he gave the Name of Moments to the momentaneous Increaſes, or infinite ſmall Parts of the Abſciſs and Area generated in Moments of Time.* He ſays, Leibnitz *hath no Symbols of Fluxions in his Method, but uſed the Symbols of Moments or Differences dx, dy, dz.* All this is ſuitable to the Doctrine of Indiviſibles. He likewiſe tells us, *becauſe we have no Ideas of infinitely little Quantities, he introduced Fluxions into his Method, that it might proceed by finite Quantities as much as poſſible.* Hence it appears, he had not at the firſt diſcovered his Doctrine of prime and ultimate Ratios, which entirely rejects Indiviſibles, or infinitely little Quantities; but at length falling upon it, *he founded his Method* (of Fluxions) *on the* primæ quantitatum naſcentium rationes, *which have a Being in Geometry, whilſt Indiviſibles, upon which the differential Method is founded, have no Being either in Geometry, or in Nature.* Accordingly he tells us, *When he is demonſtrating any Propoſition, he uſes the Letter o for a finite Moment of Time, or of its Exponent, or of any Quantity flowing uniformly, and performs the whole Calculation by the Geometry of the*

Antients, in finite Figures or Schemes without any Approximation: And so soon as the Calculation is at an End, and the Equation is reduced, he supposes, that the Moment o *decreases endlesly, and vanishes. But when he is not demonstrating, but only investigating a Proposition, for making Dispatch, he supposes the Moment* o *to be infinitely little, and forbears to write it down, and uses all Manner of Approximations, which he conceives will produce no Error in the Conclusion.* Here Sir *Isaac* declares he was wont to use the Word *Moment* in two Senses; Examples of both which he then mentions. And it is observable in his Rule for finding the Relation of Fluxions, as published out of his old Papers by Dr. *Wallis* in 1693, the Word *Moment* is used in the Sense of Indivisibles; but when he came to give that Rule himself in his Book of Quadratures first printed in 1704, he used that Word in the other Sense.

Before he had published any Thing on these Subjects, he thought fit, for the Sake of Brevity, to introduce this Term *Moment*, in the 2d Book of his *Principia Philosophiæ*. As the Geometers of his Time had been much accustomed to Indivisibles, he did not scruple there to describe *Moments* according to the Sense of that Doctrine, as he had done formerly, to be *incrementa vel decrementa momentanea*. As in another Place of that Treatise he acknowledges his using several Expressions favouring Indivisibles, but at the same Time shews how that Idea may be corrected, when such Expressions occur; so likewise here he does the like: He shews how to correct the Idea arising from this Description of *Moments*. He says, *You must never consider their Magnitudes, but their ultimate Ratio.* He adds, *it would come to the same Thing, if instead of these* Moments, *you used the Velocities of Increase or Decrease of Quantities, which he is wont to call Fluxions, or*

if you used any other finite Quantities proportional to these Fluxions.

Accordingly, the Clamour arisen against Sir *Isaac*'s Method of determining the *Moment* of a Rectangle are Mistakes, occasioned by not sufficiently attending to his last mentioned Caution. From thence it will appear, that in calling $\frac{1}{2}a$ and $\frac{1}{2}b$, the Halves of the *Moments* of A and B, he meant finite Quantities in the prime or ultimate Ratio of the correspondent Increments or Decrements of A and B. Upon this Principle, if the Sides of the Rectangle, which are denoted by A and B, be augmented and diminished by half such Lines expressed by *a* and *b*, as shall be in the ultimate Ratio of the Increments or Decrements of the Sides A and B, generated in equal Portions of Time; the Difference (aB+bA) of such Rectangles, as are contained by the Sides A and B thus augmented and diminished, will express the *Moment* of the original Rectangle.

Hence is deduced the Method of expressing the *Moments* or Fluxions of Powers, Fractions, *&c.* the practical Rules for all which, are laid down by the Authors afore mentioned. From what has been already said, and from Sir *Isaac*'s own Definition of *Moments* (in his Treatise of Fluxions and infinite Series, translated by Mr. *Colson* his present worthy Successor) *viz. The Moments of flowing Quantities, (that is, their indefinitely small Parts, by the Accession of which, in indefinitely small Portions of Time, they are continually increased) are as the Velocities of their flowing or increasing:* It is beyond all Doubt, that he did mean by that Word *Moments*, the very same Things as *Leibnitz* did by his *Infinitesimals*, in his differential Calculus, and has expressed them by the same Kind of Notation; notwithstanding all the Pains Mr. *Robins* has taken to shew they are strictly geometrical, from any subsequent Cautions Sir *Isaac* has given for correcting that Idea of them, which naturally

naturally occurs from the Definition in the last Parenthesis above. However, as their Notation is very commodious in analytic Demonstrations, and as we are taught to regard only their prime and ultimate Ratios, which are truly assignable, whether their final Magnitudes be so or not; we may be therefore permitted to use them in finding the Relation of the Fluxions of flowing Quantities, whose Relations are expressed in general by any Equation given; provided we proceed with Caution. But to return to our Purpose: The Things above premised being well understood, we come now to propose and solve a general Problem, which is the Foundation of all our Operations about Fluxions, *viz.*

An Equation being given, including any Number of flowing Quantities; 'tis required to find the Fluxions of the same.

Or in other Words, *From a fluential Equation proposed to find its proper fluxional Equation in Terms of the given One.*

By a *fluential* Equation is meant, an Equation containing flowing Quantities, whereby their Relation at all Times, and in every State, is determined: The Equation thence deduced (from the Principles hitherto laid down,) which contains Fluxions, and thereby determines the Relation of the Fluxions, is called a *fluxional* Equation. But the Solution of the Problem, and further Prosecution of the Subject, must be deferred to our next, with which we shall conclude this Work.

CONIC

CONIC SECTIONS.

The Properties of the HYPERBOLA *continued.*

PROPOSITION XVI.

As the Sum of the Transverse Axe and the Abscissa of the Ordinate from the Point of Contact, is to half the Transverse Axe, so is the Sum of the said Abscissa and the external Part, to the external Part; *that is*, BG : CA :: GT : AT. (*See Fig. to Prop.* 9.)

DEMONSTRATION.

By the 11th, $\frac{1}{2}t + x = \frac{\frac{1}{2}tx}{a}$; therefore $t + x = (\frac{1}{2}t + \frac{\frac{1}{2}tx}{a} =) \frac{\frac{1}{2}ta + \frac{1}{2}tx}{a}$, and $t + x : \frac{1}{2}t :: x + a : a$; or BG : CA :: GT : AT. Q. E. D.

PROPOSITION XVII.

As the Difference between the Transverse Axe and the external Part, is to half the Transverse Axe, so is the

the Subtangent to the Abſciſſa of the Ordinate drawn from the Point of Contact; *that is*, BT : CA :: TG : AG.

DEMONSTRATION.

By the 14th. $\frac{1}{2}t - a = \frac{\frac{1}{2}ta}{x}$, therefore $t - a = (\frac{1}{2}t + \frac{\frac{1}{2}ta}{x} =) \frac{\frac{1}{2}tx + \frac{1}{2}ta}{x}$, and $t - a : \frac{1}{2}t :: x + a : x$; or BT : CA :: TG : AG. Q. E. D.

PROPOSITION XVIII.

The Ratio of the Ordinate drawn from the Point of Contact, to the Subtangent, is equal to the Ratio compounded of the Ratios of the Diſtance between the Center and the Ordinate, to the Ordinate; and of the Parameter of the Axe, to the Axe; *that is*, $\frac{GF}{GT} = \frac{CG}{FG} \times \frac{p}{t}$.

DEMONSTRATION.

By the 9th. $tx + x^2 = \overline{\frac{1}{2}t + x} \times \overline{a + x}$; and (*by* 2*d*.) $t : p :: (tx + x^2 =) \overline{\frac{1}{2}t + x} \times \overline{a + x} : y^2$; therefore $ty^2 = p \times \overline{\frac{1}{2}t + x} \times \overline{a + x}$; whence (dividing by $x + a$ and ty) $\frac{y}{x + a} = \left(\frac{p \times \overline{\frac{1}{2}t + x}}{ty} =\right) \frac{\frac{1}{2}t + x}{y} \times \frac{p}{t}$; or $\frac{GF}{GT} = \frac{CG}{FG} \times \frac{p}{t}$. Q. E. G.

PROPOSITION XIX.

If, from the Vertices of the oppoſite Sections, and from the Center Perpendiculars be drawn to the Axe,

Axe, and cut any Tangent, and alſo an Ordinate be drawn from the Point of Contact, then theſe four Lines ſhall be proportional; *that is*, BO : CP :: FG : AQ.

DEMONSTRATION.

By the 15th. TB : TC :: TG : AT; therefore (by 4. E. 6) BO : CP :: FG : AQ. Q. E. D.

COROLLARY.

Hence, BO×AQ=CP×FG.

PROPOSITION XX.

If Perpendiculars to the Axe be drawn from the Vertices of the oppoſite Sections, and cut any Tangent; the Rectangle of theſe Perpendiculars ſhall be equal to the Rectangle of the greateſt and leaſt Diſtance of either Focus from the Vertex; *that is*, BO×AQ=KA×KB=AH×BH.

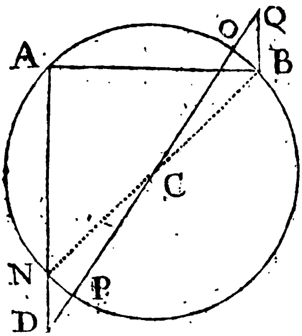
DEMONSTRATION.

Let AQ$=m$, BO$=n$, and AK or BH$=q$; then (*by ſimilar Triangles*) $m : y :: (a : x+a ::$ *by the* 16th) $\frac{1}{2}t : t+x$; alſo, $n : y :: (t-a : x+a ::$ *by* 17) $\frac{1}{2}t : x$; therefore $m=\dfrac{\frac{1}{2}ty}{t+x}$, $n=\dfrac{\frac{1}{2}ty}{x}$, and $mn=\dfrac{\frac{1}{4}t^2y^2}{tx+x^2}$; whence $mn : \frac{1}{2}t^2 :: (y^2 : tx+x^2$, *by* 2d.) $p : t$; and $mn=(\frac{1}{4}pt=$ *by* 3d) $\overline{t+q}\times q$, or BO×AQ= AK×KB=AH×BH.

LEMMA.

If a Right-line DQ, paſſing thro' the Center of any Circle, cuts two other Right-lines BQ, AD, which are drawn perpendicular to the Extremities

of any Chord AB in the said Circle, the external Parts OQ, PD of that Line shall be equal; and the Rectangle of the Perpendiculars shall be equal to the Rectangle of the Secant QP into the external Part QO.



DEMONSTRATION.

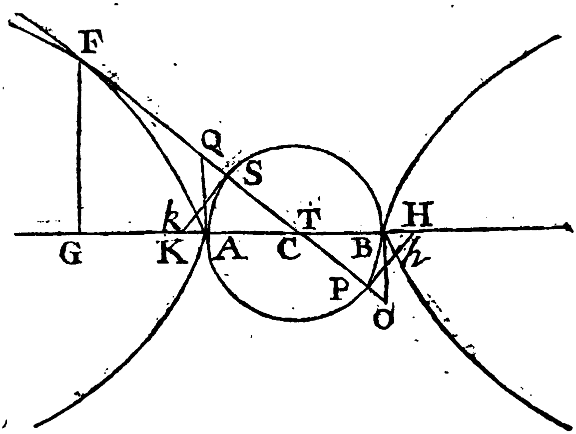
From N, the Point where the Perpendicular AD intersects the Circle, draw NB: Then it will appear that BN is a Diameter, and that the Triangles CBD and CND are similar, therefore CB : CN : : CQ : CD : : BQ : ND. But CB=CN, therefore CQ=CD, BQ=ND and QP=DO. Also (*by* 36. E. 3) DA×(DN=)BQ=DO×DP=QP×QO. Q.E.D.

PROPOSITION XXI.

If, from the Points P, S where a Circle described on the Transverse cuts any Tangent, the Lines P*h*, S*k* be drawn perpendicular to that Tangent, they will intersect the Transverse Axe produced in the focal Points K and H.

DEMONSTRATION.

The Triangles TOB, TP*h*, TAQ and TS*k* having the Angles at T common and each a Right-angle are similar;



similar; therefore BO : Pb : : Sk : AQ, and BO$\times$AQ$=$(P$b\times$S$k$$=$*by prec. Lem.*) bA$\times$$b$B, or kB$\times$$k$A. But (*by* 20) BO$\times$AQ$=HA\times$HB, or KA$\times$KB, therefore the Points K, k and H, b are coincident. Q. E. D.

COROLLARY.

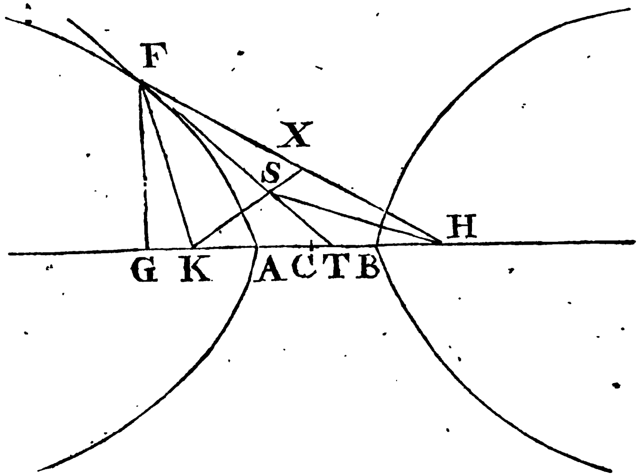
Because HA$\times$HB$=\frac{1}{4}tp$ (*by* 3d.) therefore KS$\times$PH$=\frac{1}{4}pt$.

PROPOSITION XXII.

If, from any Part of the Curve, Lines be drawn to the Foci, and the Angle formed by those Lines be bisected, then the bisecting Line will be a Tangent to the Curve in the angular Point.

DEMONSTRATION.

Take FX$=$FK, then (because by Hypothesis the Angle HFT$=$the Angle TFK) if you take any Point S, in the Line FT, the Line KS will be $=$ SX, (*by* 4. E. 1.) Draw SH, then AB (HX) $+$ (SX) SK is greater than SH, therefore the Point S is without



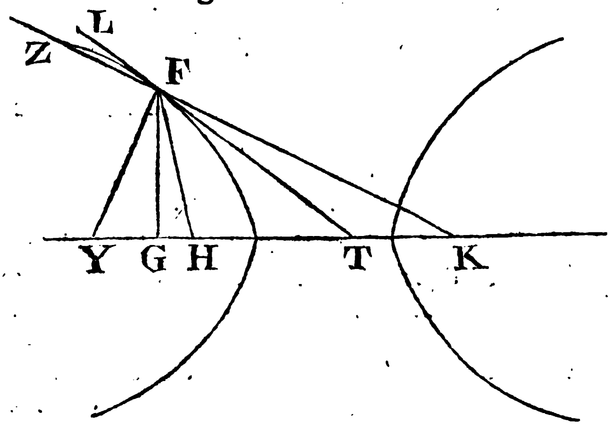
without the Curve; for, if it were in the Curve, AB+SK would be equal to SH, by the Genesis.

COROLLARY.

Hence, Lines drawn from the Foci to the Point of Contact, make equal Angles with the Tangent.

PROPOSITION XXIII.

A Right-line, drawn perpendicular to the Tangent at the Point of Contact, bisects the Angle made by Lines drawn from the Foci thro' the said Point; *that is*, if FY be perpendicular to FT, the Angle ZFY will = Angle HFY.



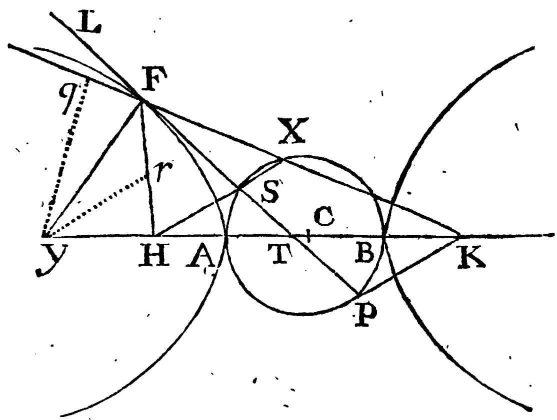
DEMON-

DEMONSTRATION.

The Angle LFY = Angle TFY by Hypothesis, and the Angle LFZ = (KFT = *by* 22) TFH, which being taken from the former leaves the Angle ZFY = Angle HFY. Q. E. D.

PROPOSITION XXIV.

If, from the Point where a Line, drawn perpendicular to the Tangent from the Point of Contact, cuts the Axe, two Lines be drawn perpendicular to the Lines which connect the Foci to the Point of Contact; the Distance in these Lines, between the Point of Contact and the Perpendiculars, will be equal to half the Parameter of the Axe; *that is*, $Fq = Fr = \frac{1}{2}p$.



DEMONSTRATION.

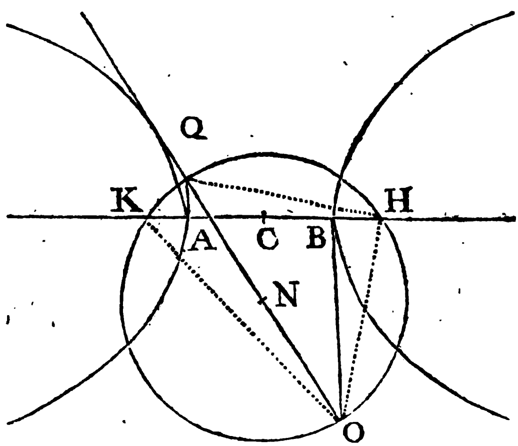
From the Points S, P where a Circle on the Transverse cuts the Tangent draw Lines to the Foci H and K, which (*by the* 21) will be perpendicular to the Tangent, then PK, HX, and YF will be parallel; continue HS to X, then (*by the* 22) HS = SX and HF = FX, therefore KX = AB = t, and the

 Triangles

Triangles KFY, KXH are similar; also, because the Angle FPK = YqF and the Angles PKF, qFY are the Compliments of the Angle qFL, the Triangles PKF, YFq are similar; therefore KX : XH :: (KF : FY ::) KP : Fq and KX × Fq = XH×KP or $\frac{1}{2}$ KX × Fq = SH ($\frac{1}{2}$XH) × KP; *that is*, $\frac{1}{2}t$× Fq = (SH×KP = *by* 21) $\frac{1}{4}pt$, or Fq=$\frac{1}{2}p$; but (*by* 26. E. 1) Fr=Fq, therefore Fq=Fr=$\frac{1}{2}p$. Q. E. D.

Proposition XXV.

If Perpendiculars from the Vertices cut any Tangent, the Part of the Tangent intercepted between the Intersections shall be the Diameter of a Circle whose Periphery shall pass thro' the Foci.



Demonstration.

By the 20th. BO × AQ = HA × HB, therefore BO : BH :: AH : AQ. But the Angle QAH = Angle OBH, therefore (*by* 6. E. 6.) the Triangles AQH, OBH are similar, and the Angle BOH = Angle AHQ. Also the Angle AQH=the Angle BHO, but the Sum of the Angles AQH, AHQ is

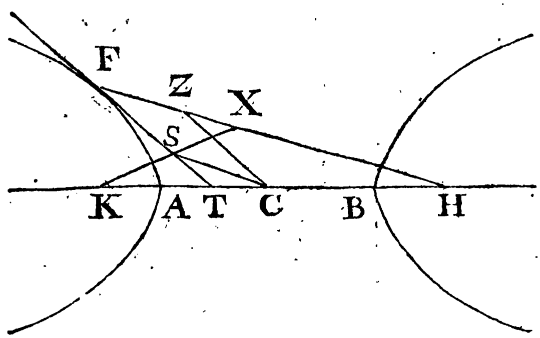
is $=$ a Right-angle; whence the Angle QHO $=$ (AHQ $+$ BHO $=$) a Right-angle, and (*by* 31. E. 3) OQ is a Diameter. In like manner QKO may be proved a Right-angle. Q. E. D.

COROLLARY.

If OQ be bisected in N, then NO$=$HN$=$NQ.

PROPOSITION. XXVI.

If from the remoter Focus, a right Line be drawn to the Point of Contact, and in that Line HX be taken$=$AB, and from the other Focus KX be drawn cutting the Tangent in S, then a right Line drawn from the Centre to that Intersection will be equal to half the transverse Axe; *that is*, CS $=$ ($\frac{1}{2}$AB) CA.



DEMONSRATION.

In the Triangles KCS, KHX, the Angle K is common, KC$=$CH, and (*by* 22) KS$=$SX; therefore (*by* 6. E. 6) the Triangles are similar and CS is parallel to HX; also CS $=$ ($\frac{1}{2}$HX $=\frac{1}{2}$AB) $=$ CB$=$ CA. Q. E. D.

PROPOSITION XXVII.

If, from the remoter Focus, a Line be drawn to the Point of Contact, and another from the Center parallel

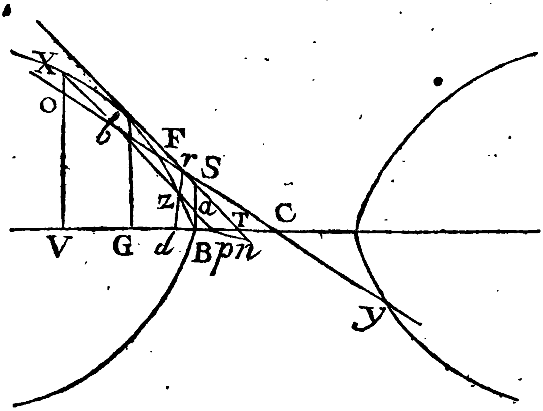
parallel to the Tangent; the Distance between the Point of Contact and Intersections of these two Lines is equal to half the transverse Axe; *that is*, $FZ = \frac{1}{2}AB$.

DEMONSTRATION.

Draw CS parallel to HF, then is the Figure FZCS a Parallelogram; therefore $ZF = (CS =$ *by* 26) $AC = \frac{1}{2}AB$. Q. E. D.

PROPOSITION XXVIII.

If, within the Curve, Lines be drawn parallel to any Tangent, they will be bisected by a Diameter produced thro' the Point of Contact. Also the Triangle BCS will be = Triangle CDr — Triangle pdz = Triangle CVo — Triangle Vpx.



DEMONSTRATION.

Put $dz = y$, $dp = c$, $Cd = n$, $BS = r$, $dr = p$, $Vx = y$, $CV = g$, $Vo = q$, $Vp = p$, and the Abscissas Bd, $BV = x$ and X respectively, then

1. By similar Triangles $\frac{y}{c} = \left(\frac{FG}{GT} = \text{by } 18.\right)$

CG

$\frac{CG}{FG} \times \frac{p}{t}$. But $\frac{CG}{GF} = \frac{\frac{1}{2}t}{r}$ by similar Triangles; therefore $\frac{y}{c} = \frac{\frac{1}{2}t}{r} \times \frac{p}{t}$, whence $\frac{\frac{1}{2}t}{r} = \frac{y}{c} \times \frac{t}{p}$ and $\frac{1}{2}tcy = ry^2 \times \frac{t}{p}$. Alſo (*by the* 2d.) $y^2 \times \frac{t}{p} = tx+x^2$, therefore $\frac{1}{2}tcy = r \times \overline{tx+x^2}$, and $rn^2 - \frac{1}{2}tcy = (rn^2 - r \times \overline{tx+x^2} =)\ r \times \overline{n^2 - \overline{tx+x^2}}$; but (*by* 6. E. 2) $n^2 - \overline{tx+x^2} = \frac{1}{4}t^2$, therefore $rn^2 - \frac{1}{2}tcy = r \times \frac{1}{4}t^2$, whence $\frac{rn^2}{\frac{1}{2}t} - cy = r \times \frac{1}{2}t$. Moreover, by ſimilar Triangles $\frac{1}{2}t : r :: n : p = \frac{rn}{\frac{1}{2}t}$, therefore (by Subſti.) $np - cy = r \times \frac{1}{2}t$, or $Cd \times dr - dp \times dz = BS \times BC$; *that is*, the Triangle Cdr — Triangle pdz = Triangle BCS.

2. By ſimilar Triangles $\frac{Y}{b} = \left(\frac{FG}{GT} = by\ 18\right)$ $\frac{CG}{FG} \times \frac{p}{t}$, and $\frac{CG}{GF} = \frac{g}{q}$, therefore $\frac{Y}{b} = \frac{g}{q} \times \frac{p}{t}$, whence $\frac{g}{q} = \frac{Y}{b} \times \frac{t}{p}$ and $gbY = qY^2 \times \frac{t}{p}$; but (*by Prop.* 2) $Y^2 \times \frac{t}{p} = tX + X^2$, therefore $gbY = q \times \overline{tX+X^2}$ and $qg^2 - gbY = (qg^2 - q \times \overline{tX+X^2} =)$ $q \times \overline{g^2 - \overline{tX+X^2}}$. Again (*by* 6. E. 2) $g^2 - \overline{tX+X^2} = \frac{1}{4}t^2$; therefore $qg^2 - gbY = q \times \frac{1}{4}t^2$, whence $qg - bY = \frac{q \times \frac{1}{4}t^2}{g}$, but (*by Sim. Trian.*) $g : q :: \frac{1}{2}t : r = \frac{q \times \frac{1}{2}t}{g}$, therefore (*by Sub.*) $qg - bY = r \times \frac{1}{2}t$, or $Vo \times CV - Vp \times Vx = BC \times BS$; that is the Triangle CVo — Tri-

F angle

angle Vpx = Triangle BCS = (by the former Part) Triangle Cdr — Triangle pdz, and (by Tranpofition) Triangle CVo — Triangle Cdr = Triangle Vpx — Triangle pdz.

3. From both Sides of the laſt Equation take the Figure $dzboV$, and the remaining Triangles oxb, bzr will be equal and ſimilar, whence $xb = bz$. Q.E.D.

Proposition XXIX.

The Triangle BSC = CFT, alſo the Trapezium $dBSr$ = Triangle pdz, Triangle bzr = Trapezium $bFTp$ and $\overline{FT+bp} \times bF = zb \times rb$.

Demonstration.

From ſimilar Triangles BS : FG :: (BC : GC :: *by* 10) CT : BC, therefore $BS \times BC = FG \times CT$; or Triangle BSC = Triangle CFT = (*by* 28) Triangle Cdr—Triangle pdz=Triangle CVo—Triangle Vpx; therefore (by Tranſpoſition) Triangle BCS+Triangle pdz=Triangle Cdr; from each Side take Triangle BCS and we have the Triangle pdz = Trapezium $dBSr$: Again from the firſt Equation Triangle $CFT+Vpx$=Triangle CVo, whence (taking Triangle CFT+Trapezium $pboV$ from each Side) we have Triangle bzr (obx) = Trapezium $bFTp$, and (*by Lem. to Prop.* 11 *of the* Parabola) $\overline{FT+bp} \times Fb = zb \times br$. Q. E. D.

Definition.

Let FS : FQ :: br : bz :: 2FT : P, the Parameter of the Diameter FY, then $P = \frac{bz \times 2FT}{br}$; and,

Proposition XXX.

As any Diameter is to its Parameter (ſo obtained) ſo is the Rectangle of the Abſciſſa into the Sum of the

the Diameter and Abſciſſa, to the Square of the Ordinate of that Abſciſſa; *that is* (putting $D=FY$, $x=Fb$, and $y=bz$, or bx) $D : P :: \overline{D+x}\times x : y^2$.

Demonstration.

By the Definition $P=\frac{y\times 2FT}{br}$, therefore $P\times\frac{\overline{D+x}\times x}{D}=\frac{y\times 2FT}{br}\times\frac{\overline{D+x}\times x}{D}=\frac{y}{br}\times\overline{D+x}\times x\,\frac{2FT}{D}$. But (*by ſim. Trian.*) $\frac{2FT}{D}=\left(\frac{FD}{\frac{1}{2}D}=\right)\frac{Tn}{(np=)x}$, therefore (by Subſtitution) $\frac{P\times\overline{D+x}\times x}{D}=\left(\frac{y}{br}\times\overline{D+x}\times Tn=\right.$ (*by Lemma to Prop.* 36. *of the Ellipſe*) $\frac{y}{br}\times y\times br=\left.\right)y^2$, therefore $D : P :: \overline{D+x}\times x : y^2$. Q. E. D.

Proposition XXXI.

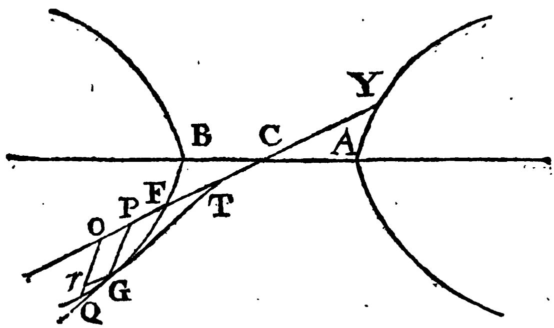
If a Tangent cut any Diameter, and from the Point of Contact an Ordinate be drawn to that Diameter, then the Sum of the Semi-diameter and Abſciſſa is to the Abſciſſa, as the Sum of the Diameter and Abſciſſa is to the Subtangent on that Diameter; *that is* $CP : FP :: YP : PT$.

Demonstration.

Let QG be an indefinitely ſmall Part of the Curve, and produced to cut the Diameter in T; draw the Ordinate GP, and, parallel to it, QO; draw Gr parallel to (YO) the Diameter continued; and put $Gr=n$, $Qr=m$, and $FT=a$, then $YP=D+x$,

 YO=

$YO=D+x+n$, $OF=x+n$, $QO=y+m$, and $PT=x+a$, also (by similar Triangles) $m:n::y:x+a$, therefore $x+a=n\times\frac{y}{m}$. But (*by Prop.* 30.) $D:$



$P::\overline{D+x+n}\times\overline{x+n}:\overline{y+m}\times\overline{y+m}$; and $D:P::\overline{D+x}\times x:y^2$, and reducing the first Analogy into an Equation, we shall have $PDx+PDn+Px^2+2Pxn-2Dym=(Dy^2=$in the 2d Analogy$)\ PDx+Px^2$, therefore $PDn+2Pxn=2Dym$, and $n=\frac{2Dym}{PD+2Px}$. Also $x+a=n\times\frac{y}{m}$, therefore $x+a=$

$$\left(\frac{2Dym}{PD+2Px}\times\frac{y}{m}=\frac{2Dy^2}{PD+2Px}=\frac{Dy^2}{p}\times\frac{2}{D+2x}=\overline{Dx+x^2}\times\frac{2}{D+2x}=\frac{2Dx+2x^2}{D+2x}=\right)\frac{Dx+x^2}{\frac{1}{2}D+x},$$

whence $\frac{1}{2}D+x:x::D+x:x+a$; or $CP:FP::YP:PT$. Q. E. D.

PROPOSITION XXXII.

The same Things being supposed as before, the Sum of the Semi-diameter and Abscissa is to the Semi-diameter, as the Semi-diameter is to the Difference

ference between the Semi-diameter and the external Part; *that is,* CP : CF :: CF : CT.

DEMONSTRATION.

CP$-$PT$=$CT. But CP$=\frac{1}{2}$D$+x$, CT$=\frac{1}{2}$D$-a$, and (*by Prop.* 31) PT $=\dfrac{Dx+x^2}{\frac{1}{2}D+x}$, therefore $\frac{1}{2}D+x-\dfrac{Dx+x^2}{\frac{1}{2}D+x}$, or $\dfrac{\frac{1}{4}D^2}{\frac{1}{2}D+x}=\frac{1}{2}D-a$; *that is,* $\frac{1}{2}D+x:\frac{1}{2}D::\frac{1}{2}D:\frac{1}{2}D-a$; or CP : CF : : CF : CT. Q. E. D.

PROPOSITION XXXIII.

As the Sum of the Semi-diameter and the Abscissa is to the Semi-diameter, so is the Abscissa to the external Part; *that is,* CP : CF : : PF : FT.

DEMONSTRATION.

By Prop. 32. $\dfrac{\frac{1}{4}D^2}{\frac{1}{2}D+x}=\frac{1}{2}D-a$, therefore $\frac{1}{4}D^2=\frac{1}{4}D^2+\frac{1}{2}Dx-\frac{1}{2}Da-xa$, and $\frac{1}{2}Da+xa=\frac{1}{2}Dx$, whence $\frac{1}{2}D+x:\frac{1}{2}D::x:a$; or CP : CF : : PF : FT. Q. E. D.

PROPOSITION XXXIV.

As the Sum of the Semi-diameter and Abscissa is to the Semidiameter, so is the Sum of the Diameter and Abscissa to the Difference between the Diameter and the external Part; *that is,* CP : CF : : YP : YT.

DEMONSTRATION.

By Prop. 33. $a=\dfrac{\frac{1}{2}Dx}{\frac{1}{2}D+x}$, therefore $D-a=$

(D$-$

$(D-\frac{\frac{1}{2}Dx}{\frac{1}{2}D+x}=)\ \frac{\frac{1}{2}D^2+\frac{1}{2}Dx}{\frac{1}{2}D+x}$ and $\frac{1}{2}D+x:\frac{1}{2}D::$ $D+x:D-a$; or CP : CF : : YP : YT. Q.E.D.

Proposition XXXV.

As the Sum of the Diameter and Abſciſſa is to the Difference between the Diameter and the external Part, ſo is the Abſciſſa to the external Part; *that is*, YP : YT : : PF : FT.

Demonstration.

By Prop. 33. $\frac{1}{2}D+x:\frac{1}{2}D::x:a$, and *(by prec.)* $\frac{1}{2}D+x:\frac{1}{2}D::D+x:D-a$, therefore *(by Equality)* $D+x:D-a::x:a$; or, YP : YT : : PF : FT. Q. E. D.

Proposition XXXVI.

As the Difference between the Semi-diameter and the external Part is to the Semi-diameter, ſo is the external Part to the Abſciſſa; *that is*, CT : CF : : FT : FP.

Demonstration.

By Prop. 32. $\frac{\frac{1}{4}D^2}{\frac{1}{2}D+x}=\frac{1}{2}D-a$, therefore $\frac{1}{4}D^2=$ $\frac{1}{4}D^2-\frac{1}{2}Da+\frac{1}{2}Dx-ax$, and $\frac{1}{2}Dx-xa=\frac{1}{2}Da$; *that is*, $\frac{1}{2}D-a:\frac{1}{2}D::a:x$, or CT : CF : : FT : FP. Q. E. D.

Proposition XXXVII.

As the Difference between the Semi-diameter and external Part is to the Difference between the Diameter and the external Part, ſo is the external Part to the Subtangent; *that is*, CT : YT : : FT : PT.

De-

DEMONSTRATION.

By Prec. $\frac{1}{2}Dx - xa = \frac{1}{2}Da$, therefore $x = \frac{\frac{1}{2}Da}{\frac{1}{2}D - a}$ and $a + x = (a + \frac{\frac{1}{2}Da}{\frac{1}{2}D - a} =) \frac{Da - aa}{\frac{1}{2}D - a}$; whence $\frac{1}{2}D - a : D - a :: a : a + x$, or CT : YT : : FT : PT. Q. E. D.

PROPOSITION XXXVIII.

As the Difference between the Semi-diameter and the external Part is to the Semi-diameter, ſo is the Difference between the Diameter and the external Part to the Sum of the Diameter and Abſciſſa; *that is*, CT : CF : : YT : YP.

DEMONSTRATION.

By Prec. $x = \frac{\frac{1}{2}Da}{\frac{1}{2}D - a}$, therefore $D + x = (D + \frac{\frac{1}{2}Da}{\frac{1}{2}D - a} =) \frac{\frac{1}{2}D^2 - \frac{1}{2}Da}{\frac{1}{2}D - a}$, and $\frac{1}{2}D - a : \frac{1}{2}D :: D - a : D + x$, or CT : CF : : YT : YP. Q. E. D.

PROPOSITION XXXIX.

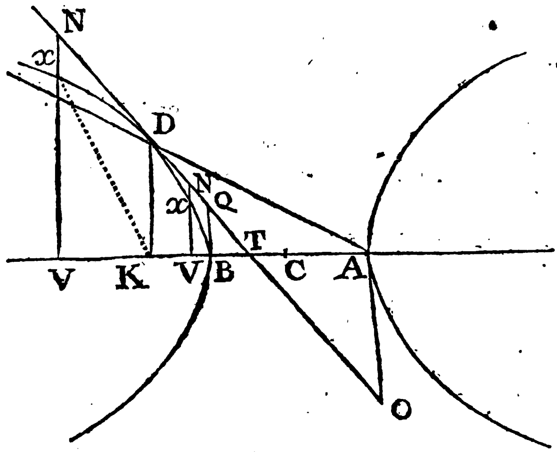
If any Ordinate to the Axe (as Vx) be continued to (N, in) the Focal Tangent (TO) then the Diſtance (VN) from the Axe to that Point in the Tangent, ſhall be equal to (Kx) the Diſtance from the Focus to the Extremity of that Ordinate.

DEMONSTRATION.

Put CK$=b$, CB$=c$, CV$=d$, then AK$=b+c$, BK$=b-c$, VK$=d \backsim b$, BV$=d-c$, and AV$=d+c$; then

1. The

1. The Point K being the Focus (*by Prop.* 4.) KL=half the Parameter of the Axe, and (*by Prop.* 3.)



CB : AK : : KB : KL, or $c : b+c :: b-c : \frac{b^2-c^2}{c}=$ (KL=) $\frac{1}{2}p$. Alſo (*by Prop.* 10.) CK : CB : : CB : CT; or $b : c :: c : \frac{c^2}{b}=$ CT; but CK—CT=KT, *that is*; $b-\frac{c^2}{b}=\frac{b^2-c^2}{b}=$ KT, and CV—CT= VT, or $d-\frac{c^2}{b}=\frac{db-c^2}{b}=$ VT, therefore (*by ſimilar Triangles*) KT : KL : : VT : VN, or $\frac{b^2-c^2}{b}$: $\frac{b^2-c^2}{c} :: \frac{bd-c^2}{b} : \frac{bd-c^2}{c}=$ VN.

2. By Prop. 2. CB : KL : : AV×VB : $\overline{Vx}^2$, or $c : \frac{b^2-c^2}{c} :: d^2-c^2 : \frac{b^2d^2-b^2c^2-c^2d^2+c^4}{c^2}=\overline{Vx}^2$,

and

and $\overline{VK}|^2 = d^2 - 2db + b^2$. But (*by* 47. E. 1.) $\overline{VK}|^2 + \overline{Vx}|^2 = \overline{Kx}|^2$; or $\frac{c^4 - 2c^2bd + b^2d^2}{c^2} = \overline{Kx}|^2$, whence $Kx = \left(\frac{bd - c^2}{c}\right.$ = by the first Part) VN. Q. E. D.

PROPOSITION XL.

If Perpendiculars be drawn from the Vertices to the Focal Tangent, then these Perpendiculars shall be equal to the Distance (in the Axe) from each Vertex to its adjacent Focus respectively; *that is*, AO=AK, and BQ=BK.

DEMONSTRATION.

By the 20. AO×BQ=AK×KB, therefore AO : AK :: KB : BQ. But (*by Prec.*) BQ=BK, therefore AO=AK. Q. E. D.

PROPOSITION XLI.

If, thro' the Point of Contact of the Focal Tangent, a Right-line be drawn to the Vertex, and any Ordinate be produced to the Tangent and cut that Line, then the Distance between the Tangent and Intersection of these Lines is equal to the Distance (in the Axe) from the Focus to the Application of the Ordinate; *that is*, DN=KV.

DEMONSTRATION.

From sim. Trian. AO : DN :: (LO : LN :: AL : LD ::) AK : KV; but AO=AK (by the Prec.) therefore DN=KV. Q. E. D.

ANSWERS
TO THE
PROBLEMS
Proposed in the Fourth NUMBER.

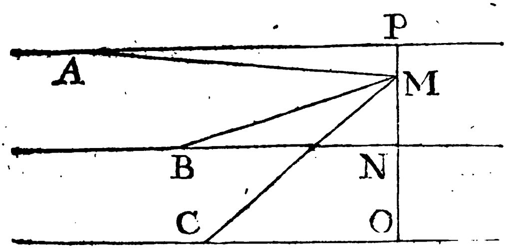
PROBLEM LIX. *Answered by* John Turner.

This Problem, in the manner it is proposed, will admit of four different Answers; but supposing that each Ship sails on the eastern Side of the Meridian departed from, the Solution will be as follows.

IN the annexed Scheme, let A represent the first Ship when in Port, B the second, and C the third; also let M represent the Place of their meeting. Then, in the Triangle BMN are given all the Angles and the Side BM, whence NB will be found $=188.407$ Miles, and $MN=78.067$, which added to, and substracted from the common Difference of Latitude gives $MO=198.067$, and $MP=41.933$.

A gain

Again, in the Triangle MOC are given the two Sides MC and MO, besides the right Angle, whence CO will be found = 104.604, the Angle MCO =

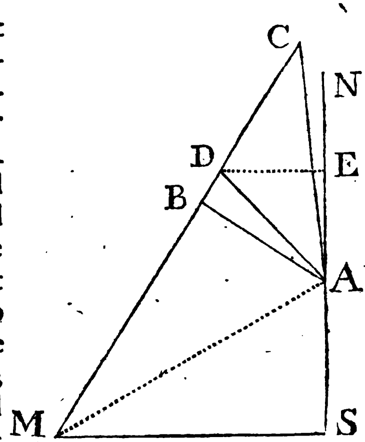


62°. 9′. 29″. and CMO = 27°. 50′. 31″. Also, in the Triangle APM are given the two Sides MA and MP besides the right Angle, whence AP will be found = 231.346, AMP = 79°. 52′. 58″. and MAP = 10°. 7′. 2″.

This Question was also answered by Mr. Moss.

PROBLEM LX. *Answered by Mr.* Moss *of* Deptford.

Let M represent *Montserrat*, A *Antigua*, MD the Distance sailed with her Starboard Tacks, DA the Distance sailed with her Larboard Tacks, B the Place where (in her first Tack) she is nearest to *Antigua*, and SN the Meridian of *Antigua*; let MD be produced to C, so that DC = DA, and join C, A.



Then in the Triangle CBA are given the two Sides BC, AB including the right

 Angle,

Angle, whence the Angle C will be found $=12^{\circ}.8'.1''$, consequently BDA$=$2BCA$=24^{\circ}.16'.3''$; therefore, it will be BA : AE :: Sine, BDA : Sine, ADE$=77^{\circ}.24'.2''$. the Cosine of the Course sailed with her Larboard Tacks aboard: Again, in the Triangle ABD are given all the ~~Angles and~~ the Side BA, whence DA the Distance sailed upon the said Tack will be found $=9.4162$: Moreover, since BDA and ADE are given, BAE the Supplement of their Sum is also given $=$ SMC $=78^{\circ}.1'.55''$. the Angle of Direction with the Wind on the first Tack: Lastly, because AMS is given, BMA will also be given and is $=10^{\circ}.31'.55''$, whence MB the Distance sailed with her Starboard Tacks aboard will be found $=20.816$ Miles.

PROBLEM LXI. *Answered by Mr.* Thomas Perryam *of* Yeovil.

Put $x+y$ for the greater Number, and $x-y$ for the lesser; then $2x^2+2y^2$ will be the Sum of the Squares, and $2x^3+6xy^2$ the Sum of the Cubes; therefore, putting $a=41$: and $b=189$, we have from the first Expression $y=\frac{1}{2}\sqrt{2a-4x^2}$, and from the last $y=\sqrt{\frac{b-2x^3}{6x}}$, therefore $\frac{1}{2}\sqrt{2a-4x^2}=\sqrt{\frac{b-2x^3}{6x}}$, whence $4x^3-3ax=-b$, and $x=4.5$, consequently the Numbers sought are 4 and 5.

The same answered by John Turner.

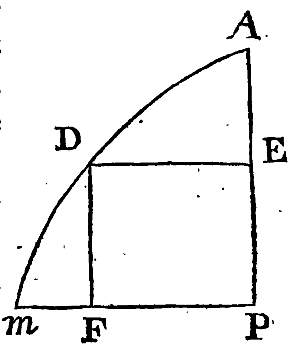
Let x represent the greater and y the lesser Number; then $x^2+y^2=(41)\ a$, and $x^3+y^3=(189)\ b$; whence $x^6+3x^4y^2+3x^2y^4+y^6=a^3$, and $x^6+2x^3y^3+y^6=b^2$, therefore $3ax^2y^2-2x^3y^3=a^3-b^2$. Put $z=$

xy,

xy, and we shall have $3az^2-2z^3=a^3-b^2$, whence $z=(xy)\,20$; the Double of which added to and taken from the first Equation gives $x^2+2xy+y^2=a+2z=81$, and $x^2-2xy+y^2=a-2z=1$, from the first of which we have $x+y=9$, and from the last $x-y=1$; whence $x=5$ and $y=4$.

This was likewise answered by Mr. Samuel Ashby *of* London.

PROBLEM LXII. *Answered by* John Turner.



Let AP$=b$, P$m=c$, the Parameter $=a$, and put AE$=x$, and ED$=y$: Then, the general Equation of the Curve being $a^{m}x^{n}=y^{m+n}$, x will be$=\dfrac{y^{\frac{m+n}{n}}}{a^{\frac{m}{n}}}$, and therefore PE $=b-\dfrac{y^{\frac{m+n}{n}}}{a^{\frac{m}{n}}}$; which being multiplied by y^2 gives $\dfrac{ba^{\frac{m}{n}}y^2-y^{\frac{m+3n}{n}}}{a^{\frac{m}{n}}}$ a Maximum, whose Fluxion $2nba^{\frac{m}{n}}y\dot{y}-\overline{m+3n}\times y^{\frac{m+2n}{n}}\dot{y}=0$; therefore

$2nba$

$2nba^{\frac{m}{n}} = \overline{m+3n} + y^{\frac{m+n}{n}}$, or $y = \overline{\frac{2nba^{\frac{m}{n}}}{m+3n}}\Big|^{\frac{n}{m+n}}$,

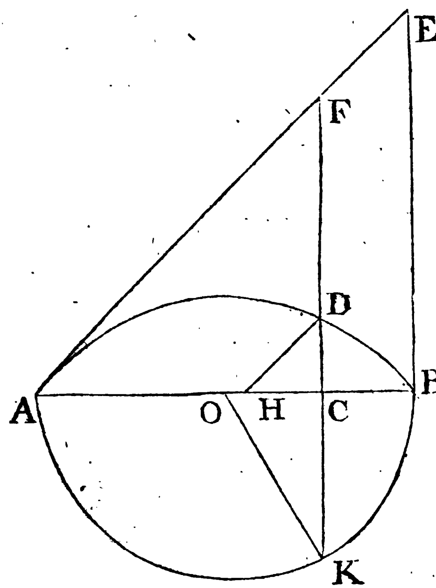
and consequently $x = \frac{2nb}{m+3n}$ and $PE = b \times \frac{m+n}{m+3n}$;

whence $PE : 2PF :: b \times \frac{m+n}{m+3n} : 2 \times \overline{\frac{2nba^{\frac{m}{n}}}{m+3n}}\Big|^{\frac{n}{m+n}}$.

Coroll. 1. If m and n be each equal to 1, the Curve will then be the common Parabola, in which $PE : 2PF :: \sqrt{b} : 2\sqrt{2a}$.

Coroll. 2. If $m = 1$, and $n = 2$, the Curve will in that Case be the Semicubical Parabola, in which $PE : 2PF :: 3b^{\frac{1}{3}} : 4a^{\frac{1}{3}} \times \sqrt[3]{14}$.

Problem LXIII. *Answered by* John Turner.



Let AB be the given Horizontal Range, AKB a Semicircle described thereon, ADB the Curve described by the Ball in its Flight, and D the Object to be struck; then by similar Triangles $BE^2 : CF^2 :: AE^2 : AF^2 ::$ (by the Laws of Descent in heavy Bodies) $BE : FD$; therefore

fore $BE^2 \times FD = CF^2 \times BE$, and $BE \times FD = CF^2$, whence $BE : CF :: CF : FD$: But $BE : FC :: AB : AC$, therefore $FC : FD :: AB : AC$, and by Division $FC : DC :: AB : BC$, also $AC : CF :: AB : BE$; whence, by compounding the two last Proportions, we have $AC : CD :: AB^2 : BC \times BE$ and

$AC \times BC = \dfrac{AB^2 + CD}{BE}$. Let DH be parallel to EA,

then $CH = \dfrac{AB \times CD}{BE}$, and therefore $AC \times BC = AB \times CH$, or $CK^2 = AB \times CH$, whence $OC = \sqrt{\frac{1}{4}AB^2 - AB \times CH}$ and $CA = \frac{1}{2}AB + \sqrt{\frac{1}{4}AB^2 - AB \times CH}$, which in the Case proposed, where $CH = CD$, gives this

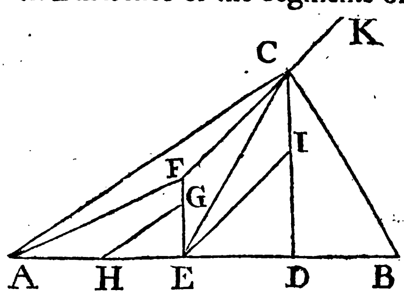
THEOREM. Find a Mean-proportional between the given Random and the Difference between one fourth of the said Random and the Object's Elevation, which add to half the given Random and the Sum will be the Distance required.

Mr. Moss *answered this Problem by an algebraic Process.*

PROBLEM LXIV. *Answered by* John Turner.

CONSTRUCTION.

In the indefinite Right line AB, take AE equal to the given Difference of the Segments of the Base;



make EF perpendicular to AB and equal to the given Difference between the Perpendicular and the

leſſer Segment, alſo make the Angle DEI equal to 45°. and draw FK parallel to EI; make EG to GH in the given Ratio of the Perpendicular to the greater Side, and thro' A draw AC, parallel to HG, meeting FK in C; draw CD parallel to FE, then join E, C and make DCB=DCE and ABC will be the Triangle required.

Demonstration.

Since the Angle DEI is $= \frac{1}{2}$ a Right-angle, and CD perpendicular to AB, it is evident that ED is = DI, therefore CD—DI (DE) = CI = (by reaſon of the Parallels) FE = the given Difference between the Perpendicular and the leſſer Segment by Conſtruction: Alſo, becauſe the Angle ECD is = BCD and CD common, it is plain that ED=DB, therefore AD—DB (ED) = AE = the given Difference of the Segments of the Baſe by Conſtruction. Moreover, by reaſon of the parallel Lines AC, HC and GE, CD it will be DC : AC :: GE : GH, *that is*, in the given Ratio of the Perpendicular to the greater Side, by Conſtr. Q. E. D.

Method *of* Calculation.

Join A, F: In the Triangle GEH are given the Sides HG and GE *(by Aſſumption)* whence the Angle H = A is given: Again, in the Triangle AEF are given the two Sides including the Right-angle, whence the Side AF and the Angle EAF will be found: Moreover, in the Triangle AFC will be given all the Angles and the Side AF, whence the Sides AC and FC = EI will be found, and conſequently $ED = \frac{EI}{\sqrt{2}}$. Then it will be AD : ED :: Tangent ACD : Tangent ECD, whence BC may be had.

Mr. Moſs *conſtructed this Problem, but in a different Manner from the preceding.*

PRO-

PROBLEM LXV. *Anſwered by Mr.* Will. Kingſton *of* Bath.

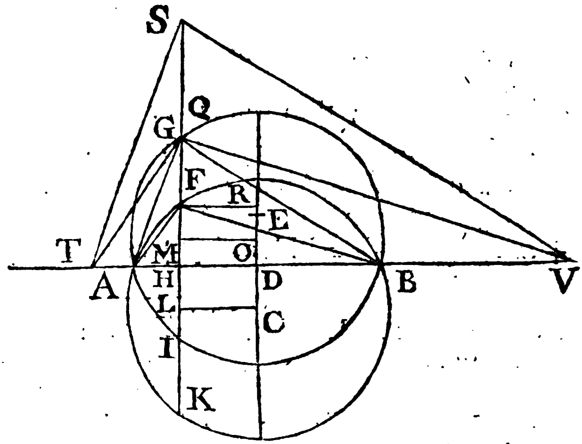
Let HF = FG = a, the Tangent of the Angle AFB = m, the Sine and Coſine of $\frac{ABG+BAG}{2}$ = s and c, and the Sine and Coſine of $\frac{BAG-ABG}{2}$ = x and y reſpectively; then will $sy+cx$ and $cy-sx$ be the Sine and Coſine of BAG and $sy-cx$ and $cy+sx$ the Sine and Coſine of ABG, whence $sy-cx : 2a :: cy+sx : \frac{2a\times\overline{cy+sx}}{sy-cx}$ = BH, and $sy+cx : 2a :: cy-sx : \frac{2a\times\overline{cy-sx}}{sy+cx}$ = AH: Alſo $a : \frac{2a\times\overline{cy-sx}}{sy+cx} ::$ 1 (Rad.): $\frac{2cy-2sx}{sy+cx}$ = the Tangent AFH, and $a : \frac{2a\times\overline{cy+sx}}{sy-cx}$ :: 1 (Rad.): $\frac{2cy+2sx}{sy-cx}$ = the Tangent BFH, whence $\frac{4sc}{s^2y^2-c^2x^2-4c^2y^2+4s^2x^2} = m$, and $my^2+4mx^2-5mc^2\times\overline{y^2+x^2} = 4sc$. But $x^2+y^2=1$, therefore $2x^2 = \frac{2\times\overline{5mc^2+4sc-m}}{3m}$ = the verſed Sine of the Difference of the Angles BAG and ABG.

The ſame anſwered by John Turner.

CONSTRUCTION.

Upon any Right-line AB let two Segments of Circles be deſcribed to contain the given Angles; join their Centers C, O by the Line CDO cutting

AB in D; take OE = OD, and DR = $\frac{2}{3}$ CE, then draw RF, parallel to AB, cutting the Periphery of the lower Circle in F; join A, F and F, B and through F draw the Right-line FHK, parallel to RC, cutting AB in H, the upper Circle in G and I and



the lower Circle in K; take HQ and QS each equal to half the given Perpendicular; draw QT and QV, parallel to FA and FB respectively, meeting AB, produced, in T and V; draw TS and VS, then TSV will be the Triangle required.

DEMONSTRATION.

Draw OM and CL, parallel to AB, meeting FK in M and L: Because of the parallel Lines, HF = DR = (*by Constr*) $\frac{2}{3}$CE = $\frac{2}{3} \times \overline{CD + 2DO}$, therefore $3HF = 2CD + 4DO$ and $4HF - 4DO = HF + 2CD$, whence $2HF - 2DO : HF + 2CD :: 1 : 2$. But $HF + 2CD = FL\ (LK) + DC = HK$, $2HF - 2DO = GM\ (MI) - DO = HI$, and by the Property of the Circle $HF \times HK\ (= AH \times HB) = GH \times HI$, therefore $GH : HF :: HK\ (HF + 2CD) : HI\ (2HF - 2DO) :: 2 : 1$. Q. E. D.

METHOD

Method of Calculation.

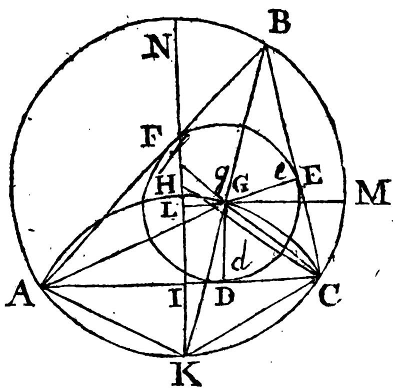
As the Cotangent of the greater Angle + twice the Cotangent of the lesser : Radius :: $\frac{3}{4}$HG : AB :: $\frac{1}{2}$ given Perpendicular to the Base of the required Triangle.

N. B. If both the Angles be obtuse, HF will be $=\frac{2}{3}\times\overline{DC-2DO}$; if both Acute, $=\frac{2}{3}\times\overline{2DO-DC}$; if one acute the other obtuse, $=\frac{2}{3}\times\overline{DC+2DO}$; if the greatest be a Right-angle, $=2DO\times\frac{2}{3}$; but, if the lesser be a Right-angle then $FH=\frac{2}{3}DC$.

Problem LXVI. *Answered by* John Turner.

Construction.

Upon any Point H, with an Interval equal to the given Radius of the circumscribing Circle, let the Circle ABCK be described, in which apply the right



Line AC equal to the given Side of the Triangle; bisect AC with the indefinite Perpendicular NK, in which take IL equal to the Radius of the inscribed Circle; through L draw LM parallel to AC, and

 upon

upon K the Point where NK intersects the Circle, with the Distance KA, describe the Arch AGC cutting LM in G; then, if through G the Line KGB be drawn meeting the Circle in B, and the Points A, B and C, B be joined, ABC will be the Triangle required.

Demonstration.

Join H, C; A, G and G, C; and draw the Perpendiculars GD, GE and GF; also bisect the Angle ACB with the Line C*g* meeting BK in *g*, and joining A, *g* draw the Perpendiculars *gd*, *gf* and *ge*.

Because AK=KC (*by Constr.*) the Angles ACK=ABK=KAC=KBC (*by Theor*: 9 *and* 10, *Book* 3. Simpson's *Geom.*) therefore (*by* 20. 1. D°.) *gf*=*ge*=*gd*: But (*by* 13. 3.) AGC+AKI=2 Right-angles, and (*by Cor.* 1. 10. 1.) AGC+CAG+ACG=2 Right-angles, therefore AKI = CAG+ACG. Now AKI+KAI (KBA) = a Right-angle= (*by Ax.* 4. 1.) KBA+ACG+CAG, therefore 2KBA+2CAG+2ACG=2Right-angles. Again 2KBA+2AC*g*+2CA*g*=2 Right-angles (*by Constr.*) therefore (*by Ax.* 1 *and* 5. 1.) 2ACG+2CAG=2AC*g*+2CA*g*, and ACG+CAG=AC*g*+CA*g*; whence (*by Ax.* 4. 1.) AGC+ACG+CAG = AGC + AC*g* + CA*g* = (*by Cor.* 1. 10. 1.) A*g*C+AC*g*+CA*g*, consequently (*by Ax.* 5. 1.) AGC=A*g*C: Whence it appears that the Points *g*, G as well as the Perpendiculars *gd*, GD; *gf*, GF; *ge*, GE coincide, and therefore GE=GF=GD = (*by* 23. 1.) IL the given Radius of the inscribed Circle by Construction. Q. E. D.

Method *of* Calculation.

In the Triangle CIH are given CH and IC, whence the Angle CHI=CBA is found; then, in the Triangle KIC are given the Right-angle I, the Angle KCI (=$\frac{1}{2}$ABC) and the Side IC, whence KC=

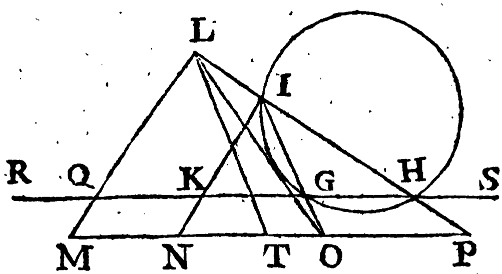
KC(=KG) and KI are had. Again, in the Triangle KGL are given the Sides KG and KL, whence the Angle GKL equal to half the Difference of the Angles BAC and BCA, will be found; and from thence the Angles themselves and the Sides AB and BC are easily found.

Mr. Samuel Clark *sent an Investigation to this Problem, whence it may be constructed.*

PROBLEM LXVII. *Answered by* John Turner.

CONSTRUCTION.

In the indefinite Right-line RS, assume GH at pleasure, upon which let a Segment of a Circle be described to contain the given Angle, and let the Line HI be inscribed therein which shall be to GH



as 2BE to AD (*Fig. to the Prob.*) make GK=GH, and join K, I and I, G; also, in HI, produced, take IL=BE and draw LM parallel to IK; then in IG, produced if needful, take IO=EF and through O draw MOP, parallel to RS, meeting LM in M and IH, produced if necessary, in P; and MLP will be the Triangle required.

DEMONSTRATION.

Since (*by Constr.*) GH : AD :: IH : 2BE, we shall have 2GH : 2AD :: IH : 2BE, or by Alternation

nation 2GH (KH) : IH :: 2AD : 2BE : : AD : BE; but by similar Triangles KH : IH : : NP : IP, therefore NP : IP : : AD : BE : : MN : IL; but IL = BE (*by Constr.*) therefore MN = AD. Q. E. D.

Method *of* Calculation.

In the Triangle HLQ are given QK, KH and LI, whence HI is found; then in the Triangle GHI will be given the two Sides GH, HI and the Angle I, whence the other Angles may be had; again, in the Triangle IPO will be given all the Angles and the Side IO, whence OP and IP are also given; then it will be PL : LM : : PI : IN.

The same answered by Mr. Moss.

Construction.

Make the Angle OIP equal to the given Angle; in PI, produced, take IL=EB, and in IO set off IO=EF; draw LT parallel to IO, and upon O, as a Center, with an Interval equal to $\frac{1}{2}$AD describe an Arch cutting LT in T; through O and T draw the Right line PM meeting LP in P; take ON= OP, join I, N parallel to which draw LM, and the Thing is done.

Demonstration.

Because OI besects NP, LT is parallel to IO and ML to NI (*by Constr.*) it appears that LT bisects MP, consequently TO=$\frac{1}{2}$MN, and MN=AD. Q. E. D.

Method *of* Calculation.

Join L, O: In the Triangle LOI are given the two Sides LI, IO with their included Angle, whence LO and the Angle ILO may be found: Again, in

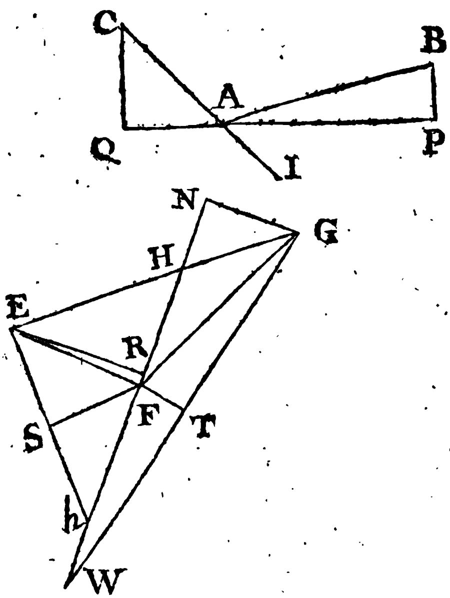
in the Triangle TLO are given the Sides TO, LO and the Angle TLO, whence the other Angles may be had: Moreover, in the Triangle IOP are given all the Angles and the Side IO, whence IP and OP become known, and consequently TP ($\frac{1}{2}$MP).

Mr. Samuel Clark *constructed this Problem from Lemma in Page* 310 Simpson's *Algebra.*

PROBLEM LXVIII. *Answered by Mr.* Tho. Moss.

CONSTRUCTION.

Let EF=AC and the Angle EFG = twice BAI (the Supplement of the given Angle) join E, G which divide in H so that GH : EH :: EF : FG



and join H, F; make the Angle CAQ = $\frac{1}{2}$EFH, then to QA, produced, draw the Perpendiculars CQ, BP and the Thing is done.

De-

Demonstration.

In HF, produced, take $Fb=EF$ and $FW=FG$; draw Eb and GW, and let ER and GN be each perpendicular to HW, FS and FT to Eb and GW respectively: Since GH : EH :: EF : FG (*by Constr.*) and GH : EH :: GN : ER (*by sim. Tria.*) it follows that EF : FG :: GN : ER and therefore $EF\times ER=FG\times GN$, that is $Fb\times ER=FW\times GN$; whence the Triangles bFE, WFG, and consequently their Halves, are equal. Also, because the Angle $CAQ=\frac{1}{2}EFH=\frac{1}{2}TbS$, the Angle $Q=S$, and the Side $AC=Fb$ (EF), it follows that the Triangles CQA and FSb are equal and alike in all respects: Moreover, the Angle BAP being $=BAI-PAI=BAI-CAQ=\frac{1}{2}EFG-\frac{1}{2}EFH=\frac{1}{2}HFG=FWT$, and $AB=WF$ (FG) the Triangles ABP and WFT are likewise equal and alike; therefore, seeing the Triangles FbS and FTW are proved equal to each other, the Triangles ACQ and ABP, which are respectively equal to them, must consequently be equal to one another. Q. E. D.

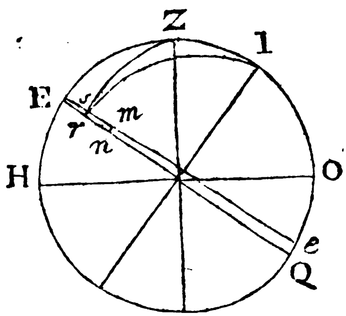
Method *of* Calculation.

In the Triangle EFG are given two Sides with their included Angles EFG, whence EG and the Angle FEG may be found; then, from the Part EH being known, there will be given two Sides and their included Angle FEH, from whence the Angle EFH (2CAQ) may be had.

Problem LXIX. *Answered by Mr.* Moss.

Let EQ be the Equinoctial, E the Sun in the Meridian, and Ese the Line described by the Sun's apparent Motion in half a Revolution, to which from the Zenith (Z) let an Arch of a great Circle be drawn, meeting EQ in r; then, it is evident that

Zs is the Sun's nearest Distance to the Zenith, and that the Arch Er will express the Time from Noon, answering to the greatest Altitude; to determine

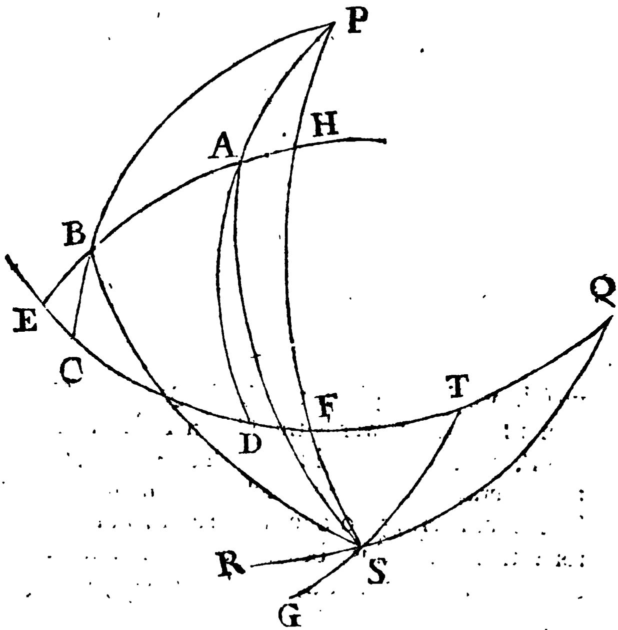


which we have given the Arch Qe, expressing the Sun's Progress towards the North Pole in 12 Hours $=11'.\ 50''$, which divided by 180 gives $3''.944$ for the Alteration (mn) of Declination in four Minutes Time, answering to one Degree (En) of the Equator; therefore, this, because of its Smallness, being considered as rectilineal, we shall have $3600''$ (En) : $3''.944$ (mn) :: Radius : Tangent of the Angle nEm, which the apparent Path of the Sun makes with the Equinoctial, $= 3'.\ 46''$, which taken from 90°. leaves $89^{\circ}.\ 51'.\ 14''. =$ the Angle ZEs: Whence Radius : *Cos*. ZEs :: Tangent ZE : Tangent Es (or Er) $= 4'.\ 44''$, answering to $18''.\ 56'''$. the Time required.

PROBLEM LXX. *Answered by Mr.* Moss.

Let P represent the North Pole, EQ an Arch of the Equator, S the Sun at the Time required, PBC and PAD the Meridians of the two Places B and A, GST and RSQ their respective Horizons, and PFS

a Meridian passing through the Sun; then the Arch C D (Angle CPD) will represent the Difference of Longitude. Now it is evident that BS and AS are



each $= 90°$, and that the Angles made with the Equator by the respective Horizons will be equal to the Complements of the Latitudes of the two Places respectively; therefore in the Triangle STQ are given the Angles SQT, STQ and the Side TQ (=CD) whence ST is known; and in the right-angled Triangle SFT are given the Angle STF (BP) and the Side ST, whence SF the required Declination may be found.

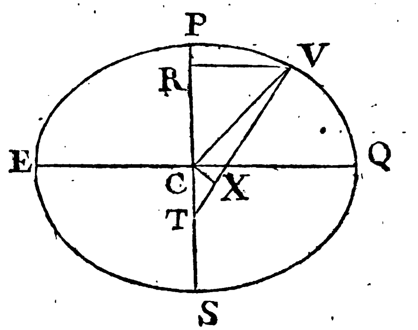
The same answered by John Turner.

Let S represent the Sun at the Time required, B and A the two given Places, and P the North Pole: Then

Then in the Triangle PBA are given the two Sides PB, PA with their included Angle to find the Angle PAB, which being known its Supplement PAH will likewise be known; therefore in the Triangle PAH, right angled at H, are given the Side PA and the Angle A to find the Side PH the Declination required.

PROBLEM LXXI. *Answered by* John Turner.

Let the Square of CQ be to that of CP as $1+B$ to 1, and put CR$=x$; then, by the Property of the



Curve, VR$=\sqrt{\overline{1+B}\times\overline{1-x^2}}$, and RT$=\overline{1+B}\times x$, therefore CV$=\sqrt{1+B-Bx^2}$ and TV$=\sqrt{\overline{1+B}\times\overline{1+Bx^2}}$; whence, by similar Triangles, TV : RV :: TC : CX$=\dfrac{Bx\sqrt{1-x^2}}{\sqrt{1+Bx^2}}$, therefore CV : CX :: Radius : Sine of CVX$=\dfrac{Bx\sqrt{1-x^2}}{\sqrt{1+Bx^2}\times\sqrt{1+B-Bx^2}}$, which by the Question must be a Maximum, consequently its Fluxion $\dfrac{B\dot{x}\sqrt{1-x^2}}{\sqrt{1+Bx^2}\times\sqrt{1+B-Bx^2}}-$

$$\frac{Bx^2\dot{x}}{\sqrt{1-x^2}\times\sqrt{1+Bx^2}\times\sqrt{1+B-Bx^2}}-$$

$$\frac{B^2x^2\dot{x}\quad\sqrt{1-x^2}}{\overline{1+Bx^2}|^{\frac{3}{2}}\times\sqrt{1+B-Bx^2}}+$$

$$\frac{B^2x^2\dot{x}\sqrt{1-x^2}}{\sqrt{1+Bx^2}\times\overline{1+B-Bx^2}|^{\frac{3}{2}}}=0;$$ whence $1-$

$$x^2+\frac{Bx^2\times\overline{1-x^2}}{1+B-Bx^2}=x^2+\frac{Bx^2\times\overline{1-x^2}}{1+Bx^2}$$ and $x=\sqrt{\frac{1}{2}}$;

therefore RC $=.7071067$, RT $=.7133258$, RV $=.7101241$, CV $=1.0021857$, VT $=1.00652$, CX $=.0043878$, RVT $=45^\circ.7'.39''$, VTR $=44^\circ.52'.21''$, and CVX $=15'.8''$.

PROBLEM LXXII. *Answered by* John Turner.

Let the Cosine of the lesser Latitude be denoted by c, that of the greater by C (Radius being Unity) and let the required Sine of the Declination be de-denoted by d and its Cosine by p; then (*by Prob.* 15. *Pa.* 180. Simpson's *Fluxions*) the two principal Diameters of the Ellipsis described in the former Latitude will be $\frac{2pd}{d^2-c^2}$ and $\frac{2p}{\sqrt{d^2-c^2}}$, and those in the latter $\frac{2pd}{d^2-C^2}$ and $\frac{2p}{\sqrt{d^2-C^2}}$; therefore, putting $.7854=m$, we have $\frac{4mp^2d}{\overline{d^2-c^2}|^{\frac{3}{2}}}=$ (320 Square Poles) the Area of the Ellipsis described in the lesser Latitude, and $\frac{4mp^2d}{\overline{d^2-C^2}|^{\frac{3}{2}}}=$ (160 Square Poles) the

Area

Area of the Ellipsis described in the greater Latitude. Hence $\overline{d^2-C^2}|^{\frac{3}{2}} : \overline{d^2-c^2}|^{\frac{3}{2}} :: 2 : 1$, or $d^2-C^2 : d^2-c^2 :: 2^{\frac{2}{3}}\ (n) : 1$, therefore $d^2-C^2=nd^2-nc^2$, $d^2=\frac{nc^2-C^2}{n-1}$, and $p^2\ (=1-d^2)=\frac{n-1-nc^2+C^2}{n-1}$; which Values being substituted in the former Expressions of the Area, or $\frac{p^2 d}{\overline{d^2-c^2}|^{\frac{3}{2}}}=\frac{320}{4m}$, gives

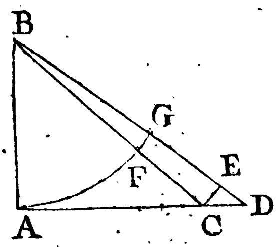
$$\frac{\overline{n-1-nc^2+C^2}\times\sqrt{nc^2-C^2}}{\overline{c^2-C^2}|^{\frac{3}{2}}}=\frac{320}{4m}, \text{ whence}$$

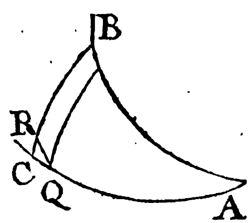
$$\frac{\overline{n-1-nc^2+C^2}|^2\times\overline{nc^2-C^2}}{\overline{c^2-C^2}|^3}=\overline{\frac{320}{4m}}|^2=10375.3,$$

the lesser Latitude $=74^\circ.\ 54'.\ 26''$, the greater Latitude $=76^\circ.\ 14'.\ 26''$, and Declination $=17^\circ.\ 8'.\ 18''$.

Problem LXXIII. *Answered by* John Turner.

In the spherical Triangle ABC let AB represent the Equator and AC the Ecliptic; then, putting the





Sine of A $=d$, its Cosine $=p$, and the Tangent of BC (Declin.) $=x$, we shall have the Sine of BC $=$

x

$\frac{\dot{x}}{\sqrt{1+x^2}}$. But $d : \frac{x}{\sqrt{1+x^2}} :: 1 : \frac{x}{d\sqrt{1+x^2}}$ = the Sine of AC, therefore its Tangent $= \frac{x}{\sqrt{d^2-p^2x^2}}$. Let now, in the right-angled plain Triangle ABC, AB $=1$, AC $=t$, the Arch AF $=z$, its Fluxion FG $=\dot{z}$, and CD $=\dot{t}$; then $1 : \dot{z} :: \sqrt{1+t^2} : \dot{z}\sqrt{1+t^2} =$ CE, and $1 : \sqrt{1+t^2} :: \dot{z}\sqrt{1+t^2} : \dot{z}\times\overline{1+t^2} = \dot{t} =$ CD.

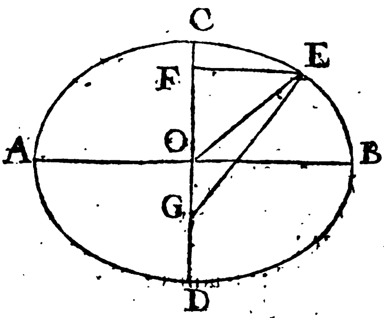
But in the spherical Triangle, if L be put for the Tangent of the Sun's Longitude from *Aries*, we shall have Tangent AC : Tangent BC :: Radius : Cosine ACB :: Tangent QC : Tangent CR :: $\dot{L} : \dot{z}$; therefore $\dot{z} = \dot{L}\times\sqrt{d^2-p^2x^2}$, consequently $\dot{t} = \dot{L}\times\sqrt{d^2-p^2x^2}\times\overline{1+t^2}$. Moreover, if the Tangent of Height of the Equator be put $=m$, the Tangent of the Sun's meridional Altitude will be $\frac{m-x}{1-mx}$, therefore its Cotangent $t = \frac{1-mx}{m-x}$; whence our Expression will be $\dot{L}\times\sqrt{d^2-p^2x^2}\times\overline{1+\left(\frac{1-mx}{m-x}\right)^2} =$ a Maximum, therefore its Fluxion $=0$, or $p^2m^2x^4 - 3m^3p^2x^3 - \overline{2md^2-2m^3d^2+mp^2}\times x^2 + \overline{4m^2p^2-p^2}\times x = 2m^2d^2-2d^2$, whence x may be found.

PROBLEM LXXIV. *Answered by* John Turner.

Let OB $=a$, CO $=b$, and OF $=x$; then by the Property of the Curve $CO^2 : OB^2 :: FC\times FD : FE^2 = \frac{\overline{b^2-x^2}\times a^2}{b^2}$, whence FE $= \frac{a}{b}\sqrt{b^2-x^2}$, and (putting

ting $d = -\frac{b^2+a^2}{b^2}$) $OE = \sqrt{a^2 - dx^2}$.

Therefore, the Gravity at the Surface of a Spheriod,



differing but little from a Sphere, being inversely as the Distance from the Center very nearly, the Force with which the Body descends from the Point E will be to the Force with which it descends from the Point B, as OB to OE, or as $a : \sqrt{a^2 - dx^2}$.

Moreover, the Force in either Case, as the Body approaches the Center, decreasing directly as the Distance, the Time of Descent will be equal to one fourth of the Time of Revolution in the Circle whose Radius is OA or OE.

Let, therefore, the Time of Revolution at the Equator, which is known to be 1 H. 24 M. 36 S. be denoted by T; then, since the Periodic Times in Circles are universally in the subduplicate Ratio of the centripetal Forces inversely and the subduplicate Ratio of the Radii directly, we have $\frac{\sqrt{OB}}{\overline{a^2-dx^2}|^{\frac{1}{4}}} : \frac{\sqrt{OE}}{\sqrt{a}} ::$

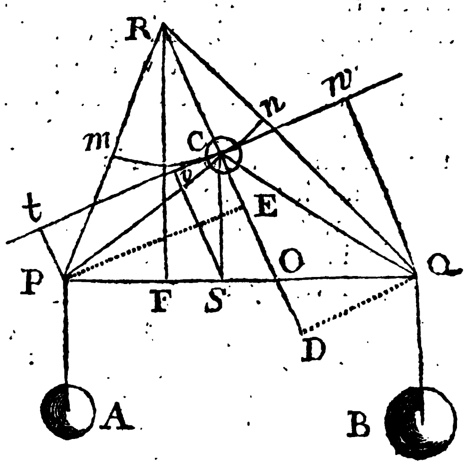
$T : \frac{T\sqrt{OE} \times \overline{a^2-dx^2}|^{\frac{1}{4}}}{a}$ the Time of Revolution in the Circle whose Radius is OE with the centripetal Force

Force at B: Therefore $\frac{3}{4}T - T \times \frac{\overline{a^2-dx^2}^{\frac{1}{2}}}{4a}$ is equal to the Difference of the Times of Descent along BO and EO$=t$ the given Time, whence $a - \sqrt{a^2-dx^2} = \frac{4at}{T}$, and $x = \frac{2a}{T} \times \sqrt{\frac{2tT - 4tt}{d}}$; therefore OF and FE are found equal to 1716.5 and 3591.7 respectively.

Then, by the Property of Ellipsis, $OC^2 : OB^2 ::$ FO : FG (GE being perpendicular to the Surface at E) and GF : FE :: Radius : Tangent of FGE the Complement of the required Latitude, whence the Latitude itself is $= 45^\circ.\ 14'$.

PROBLEM LXXV. *Answered by* John Turner.

Suppose the Weight *C* to move in the Arch *mn*, towards *m*, with a Velocity expressed by Unity; let



RC, produced, meet PQ in O, and let CS be perpendicular to PQ; then the Velocity of C in the perpendicular Direction CS, being to the absolute Velocity as Radius to the Cosine of tCS (or the Sine of the Angle SCO), will be expressed by the Sine of SCO, therefore the Momentum of the Weight C in the Direction perpendicular to the Horizon is $C\times$SCO. Moreover, the Velocity of C in the Direction CP (or the Velocity of A in the Direction of the Horizon) being as the Cosine of tPC (or the Sine of PCO) will be expressed by the Sine of the Angle PCO, and the Momentum of the Weight A towards the Horizon will be $A\times$ Sine PCO, In like manner the Momentum of B from the Horizon will be expressed by $B\times$ Sine of the Angle QCO. But, when the Weights are in Equilibrio, the Momentum of the two former must be equal to that of the latter, or $C\times$ Sine SCO$+A\times$Sine PCO$=B\times$Sine QCO.

Now, in order to get an Expression for this in algebraic Terms, draw RF perpendicular to PQ, and PE and QD perpendicular to RO, produced; then, putting RP$=a$, RC$=d$, RQ$=c$, and the Tangent of FRO$=x$, we shall have the Sine of FRO$=$

$\frac{x}{\sqrt{1+x^2}}$ and its Cosine $=\frac{1}{\sqrt{1+x^2}}$; therefore, if the

Sines of QRF and PRF be denoted by n and m and their Cosines by p and q respectively, the Sine of QCO

will be found to be $=\frac{c\times\overline{n-px}}{\sqrt{\overline{c^2+d^2}\times\overline{1+x^2}-2dc\times\overline{p+nx}}}$,

that of PCO $=\frac{a\times\overline{m+qx}}{\sqrt{\overline{a^2+d^2}\times\overline{1+x^2}-2da\times\overline{q-mx}}}$

and that of OCS $=\frac{x}{\sqrt{1+x^2}}$: Whence $A\times$

$$\frac{a\times\overline{m+qx}}{\sqrt{\overline{a^2+d^2}\times\overline{1+x^2}-2da\times\overline{q-mx}}}+C\times\frac{x}{\sqrt{1+x^2}}=$$

$$B\times\frac{c\times\overline{n-px}}{\sqrt{\overline{c^2+d^2}\times\overline{1+x^2}-2dc\times\overline{p+nx}}}.$$

COROLL. 1. If $C=0$, then $A\times$ Sine PCO $=B\times$ Sine QCO, which is the Case of the Ballance of unequal Brachia; but, if $A=B$, then it will become the same as the common Balance, and RO will bisect PCQ.

COROLL. 2. If $B=0$, then $A\times$ Sine PCO $=-C\times$ Sine SCO, in which Case C will be on the other Side of the Perpendicular RF; therefore if $A=C$, the Line CO will bisect the Angle PCS.

COROLL. 3. But if $A=0$, then $C\times$ Sine SCO $=B\times$ Sine QCO; therefore when $C=B$, the Angle SCO will be $=$ the Angle QCO.

PROBLEM LXXVI. *Answered by* John Turner.

In order to give a Solution to this Problem, it will be necessary to premise the following

LEMMA.

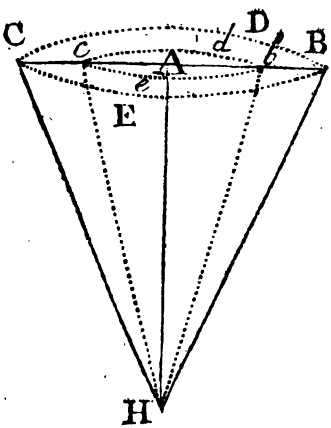
If a Corpuscle, posited in a Line, passing thro' the Center, perpendicular to the Plane of the Circle BECD, be attracted by every Particle of Matter in the Surface of that Circle with Forces that are as the n Power of the Distance; then the Attraction of the whole Plane, or the Force with which the said Corpuscle is impelled in the Direction HA will be expressed by

$$\frac{2p\times AH\times BH^{n+1}-2p\times AH^{n+2}}{n+1}.$$

For put AH$=a$, H$b=x$, and suppose the Circle *becdb* to increase continually 'till it coincides with BECDB; then, if p be put $=3.1416$, *&c.* because

3 Hb^2

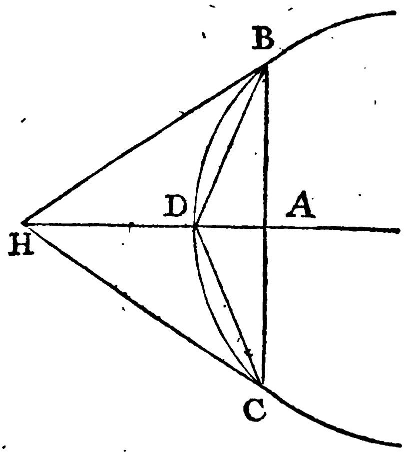
$Hb^2 - AH^2 = Ab^2$ is $= x^2 - a^2$, we have $p \times \overline{x^2 - a^2}$ for the Area *becdb*, whoſe Fluxion $2px\dot{x}$ is as the Quantity of Matter or the Number of Particles acting



on the Corpuſcle at H in Directions ſimilar to H*b*. Now the abſolute Force of a Particle in the Direction H*b* being ſuppoſed as the *n* Power of the Diſtance, or expreſſed by x^n, we ſhall have by the Reſolution of Forces H*b* (x) : HA (a) : : x^n (the Force in the Direction H*b*) : ax^{n-1}, the Force of the ſame Particle in the Direction HA; which multiplied by $(2px\dot{x})$ the Number of Particles acting in ſimilar Directions, gives $2pax^n\dot{x}$ for the Fluxion of the whole Force in the Direction HA, whoſe Fluent is $\frac{2pax^{n+1} - 2pa^{n+2}}{n+1}$; which, when H*b* (x) becomes $=$ HB, will be $\frac{2p \times AH \times HB^{n+1} - 2p \times AH^{n+2}}{n+1}$. Q.E.I.

 Let

Let BDC represent a Curve of any kind, in which $AD=x$, and $AB=y$; then, putting $HD=a$, the Attraction of the circular Plane BAC, by the preceding



Lemma will be $\frac{2p\times AH\times HB^{n+1}-2p\times AH^{n+2}}{n+1}$

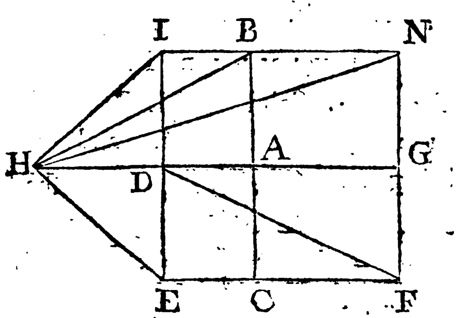
$$=\frac{2p\times\overline{a+x}\times\overline{\overline{a+x}^2+y^2}\big|^{\frac{n+1}{2}}-2p\times\overline{a+x}\big|^{n+2}}{n+1}:$$

Therefore, if this Expression be multiplied by $\dot{x}$ and $2p$ be rejected (it being a constant Quantity) we shall have $\frac{\dot{x}\times\overline{a+x}\times\overline{\overline{a+x}^2+y^2}\big|^{\frac{n+1}{2}}-\dot{x}\times\overline{a+x}\big|^{n+2}}{n+1}$ for the general Expression of the Fluxion of the required Attraction; whose Fluent, being found by substituting for x or y their Values given from the Nature of the Curve, will be the Attraction required.

Coroll. If $a=0$, or H coincides with D, the Expression will become $\frac{x\dot{x}\times\overline{x^2+y^2}|^{\frac{n+1}{2}}-x^{n+2}\dot{x}}{n+1}=$ the Fluxion of the Attraction upon a Corpuscle at the Vertex; from whence and the Nature of the Curve the Attraction in any Solid may be found.

Example I.

Let the given Solid FEIN be a Cylinder; then y being a constant Quantity, or $=b$, the general Expres-



sion will become $\frac{\dot{x}\times\overline{a+x}\times\overline{\overline{a+x}|^2+b^2}|^{\frac{n+1}{2}}-\dot{x}\times\overline{a+x}|^{n+2}}{n+1}$;

whereof the corrected Fluent is $\frac{\overline{\overline{a+x}|^2+b^2}|^{\frac{n+3}{2}}-\overline{a^2+b^2}|^{\frac{n+3}{2}}+a^{n+3}-\overline{a+x}|^{n+3}}{\overline{n+3}\times\overline{n+1}}=$ the Attraction of the Cylinder BCEI; which, when DA=DG, will become

HN

$$\frac{HN^{n+3}-HI^{n+3}-HG^{n+3}+HD^{n+3}}{\overline{n+3}\times\overline{n+1}} = \text{the Attraction}$$

of the whole Cylinder, upon a Corpuscle at H.

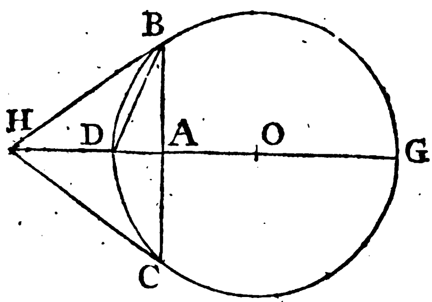
COROLL. 1. If H coincide with D, or $a=0$, the Expression will become $= \frac{DF^{n+3}-FG^{n+3}-DG^{n+3}}{\overline{n+3}\times\overline{n+1}}$ the Attraction of the Cylinder upon a Corpuscle in the Vertex.

COROLL. 2. If $n=-2$, or the Force be inversely as the Square of the Distance, the Attraction upon a Corpuscle at H will be as HI+IN—HN; and at D, as IN+GN—DF.

EXAMPLE II.

Let the given Solid GBDC be a Sphere, and put the Radius OD $=b$, and AO $=c$; then y being $=\sqrt{2bx-x^2}$, we shall have

$$\frac{\dot{x}\times\overline{a+x}\times\overline{\overline{a+x}^2+2bx-x^2}^{\frac{n+1}{2}}-\dot{x}\times\overline{a+x}^{n+2}}{n+1}$$ for the



Fluxion of the Attraction; whereof the corrected Fluent is $\frac{a^{n+3}-\overline{a+x}^{n+3}}{\overline{n+1}\times\overline{n+3}}+\frac{\overline{n+3}\times\overline{a^2+2cx}^{\frac{n+5}{2}}-a^{n+5}}{2c^2\times\overline{n+1}\times\overline{n+3}\times\overline{n+5}}+$

$n+$

$$\frac{\overline{n+5}\times\overline{\overline{2ca-a^2}\times\overline{a^2+2cx}}\,\big|^{\frac{n+3}{2}}-a^{n+3}}{2c^2\times\overline{n+1}\times\overline{n+3}\times\overline{n+5}};$$

which, when $x=2b$, will become $=$

$$\frac{HG^{n+3}\times\overline{n+1}\times\overline{ab+b^2-a^2}+HD^{n+3}\times\overline{n+5}\times\overline{ab+b^2+a^2}}{c^2\times\overline{n+1}\times\overline{n+3}\times\overline{n+5}}$$

the Attraction of the whole Globe.

Coroll. 1. If H be taken at D, or the Attraction of the Sphere upon a Corpuscle in its Surface be required, HD or a being then $=0$, the Expression will become $\dfrac{DG^{n+3}}{\overline{n+3}\times\overline{n+5}}+\dfrac{0^{n+3}}{\overline{n+1}\times\overline{n+5}}$, which when $n+3$ is an affirmative Number will be as $\dfrac{DG^{n+3}}{\overline{n+3}\times\overline{n+5}}$; but otherwise 0^{n+3} will come into the Denominator, and the Value sought will be infinite.

Coroll. 2. If $n=-2$, or the Attraction be inversely as the Square of the Distance, we shall have $\dfrac{2b^3}{3c^2}\left(\dfrac{2OD^3}{3HO^2}\right)$ for the Attraction at H, and $\dfrac{2b}{3}$ $\left(\dfrac{2OD}{3}\right)$ for that at D.

Coroll. 3. Hence, it is manifest that the Attraction of any other Globe, whose Radius is B, and the Distance of the Corpuscle from the Center C, will be expressed by $\dfrac{2D^3}{3C^2}$; therefore the Attraction of the two Globes, whose Radii are b and B at the Distance c and C, are to each other as $\dfrac{2b^3}{3c^2}$ to $\dfrac{2B^3}{3C^2}$; or as the Quantities of Matter applied to the Squares of their respective Distances.

Coroll.

Coroll. 4. Therefore, if the Bodies be equal, or the Attraction of the same Body at different Distances be required; then the Proportion will be barely as $\frac{1}{c^2} : \frac{1}{C^2}$, or as $C^2 : c^2$; that is as the Square of the Distance from the Center inversely.

Coroll. 5. But, if the Bodies be unequal, and the Attraction at equal Distances from the Center be required; the Proportion will become as $b^3 : B^3$; that is as the Bodies themselves.

Coroll. 6. If the Distances from the Centers be proportional to the Diameters of the Spheres respectively; then, because $C^2 = \frac{B^2c^2}{b^2}$, if $\frac{B^2c^2}{b^2}$ be substituted in the Proportion in Corollary 3, it will become as $b : B$; that is the Attraction in this Case will be as the Diameters or Radii.

Coroll. 7. It appears, that the Attraction of any spherical Body will be the same on a Particle without its Surface, as if the whole Quantity of Matter in that Sphere was contracted into a single Corpuscle placed in its Center.

Coroll. 8. Hence, if instead of a single Particle, we suppose another Globe at any Distance from the Center O; then because each Particle in the said Globe is attracted by the Matter in BGCD the same as if it was all contracted in the Center O; and every Particle in BGCD, in like manner by that other Globe; it follows, therefore, that the absolute Force with which two spherical Bodies tend to each other, is as the Product of their Quantities of Matter applied to the Square of the Distance of their Centers. For the Attraction of the Globe BGCD upon a single Particle at H being expressed by $\frac{2b^3}{3c^2}$, let this Expression be multiplied by B^3, or the Number of

Particles

Particles of the Globe whose Center is the Point H, and Radius B, and there will arise $\frac{2b^3B^3}{3c^2}$ for the Ratio of the Attraction exerted by the Globe GBDC upon the Globe whose Center is H; which Expression, is manifestly as the Quantity of Matter in the Globes applied to the Square of their Distance.

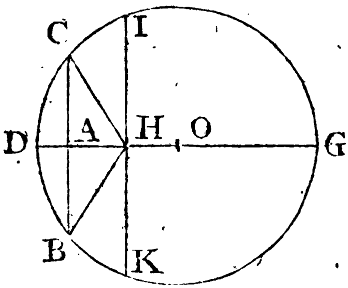
COROLL. 9. If r and R be the Radii of two other Globes, whose Centers are at the same Distance from each other, as those of the two former; then because the Ratio of their Attraction upon each other, by the precedent, appears to be $\frac{2r^3R^3}{3c^2}$, $\frac{2b^3B^3}{3c^2}$ will be to $\frac{2r^3R^3}{3c^2}$ as $b^3B^3 : r^3R^3$; *that is*, the accelerating Attraction of any two Globes towards each other, is to that of any other two Globes, whose Centers are at the same Distance from each other as those of the former, as the Product of two the Quantities of Matter in the two former, to the Product of the Quantities of Matter in the two latter.

COROLL. 10. But, when the Distance of the Centers of the two Globes, whose Radii are r and R, is not equal to that of the other two; then, calling that Distance C, the Ratio of the Attractions will be as $\frac{2r^3R^3}{3C^2} : \frac{2b^3B_3}{3c^2}$; that is as the Product of the Quantities of Matter in the Spheres, applied to the Square of the Distance between their Centers.

EXAMPLE III.

Let the Point H be supposed with in a Sphere; then, DH being $= a$, and the rest as before, we have $\frac{\dot{x}\times\overline{a-x}\times\overline{\overline{a-x}|^2+2bx-x^2}|^{\frac{n+1}{2}}-\dot{x}\times\overline{a-x}|^{n+2}}{n+1}$ for the

the Fluxion of the Segment BDC; whereof the corrected Fluent is $\dfrac{a^{n+5} - \overline{a^2 + 2cx}|^{\frac{n+5}{2}}}{2c^2 \times \overline{n+1} \times \overline{n+5}}$



$$\frac{\overline{+2\overline{ab-a^2}\times\overline{a^2+2cx}|^{\frac{n+3}{2}}-a^{n+3}}}{2c^2\times\overline{n+1}\times\overline{n+3}}+\frac{\overline{a-x}|^{n+3}-a^{n+3}}{\overline{n+1}\times\overline{n+3}};$$

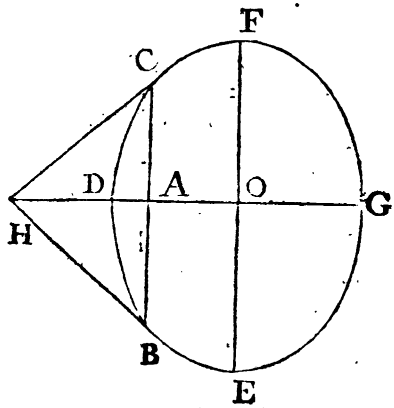
which, when $x=a$, will (by putting $HI=d$) become $=$ $\dfrac{DH^{n+3}\times\overline{n+5}\times\overline{ba-b^2-a^2}+HI^{n+3}\times d^2}{c^2\times\overline{n+1}\times\overline{n+3}\times\overline{n+5}}$ the Attraction of the Segment KDI. In like manner the Attraction of the Segment KGI will be found to be $=$ $\dfrac{GH^{n+3}\times\overline{n+5}\times\overline{br-b^2-r^2}+HI^{n+3}\times d^2}{c^2\times\overline{n+1}\times\overline{n+3}\times\overline{n+5}}$, r being $=$ GH, the rest as before; therefore the Attraction tending towards the Center will be the Difference of these two Expressions.

COROLLARY. If the Law of Attraction be inversely as the Square of the Distance, we shall have $\dfrac{GH\times\overline{3br-3b^2-r^2}-DH\times\overline{3ba-3b^2-a^2}}{-3c^2}$ for the Attraction of the Corpuscle towards the Center, which

which reduced becomes $=\frac{2c}{3}$, that is as the Diſtance from the Center.

EXAMPLE IV.

Let the given Solid BEGFC be a Spheriod, wherein $DG=2b$, $EF=2c$, $DA=x$ and $CA=y$; then, by the Property of the Curve $y^2=\frac{c^2}{b^2}\times\overline{2bx-x^2}$, and therefore



$$\frac{\frac{\overline{b^2-c^2}|^{\frac{n+1}{2}}}{b^{n+1}}\times\dot{x}\times\overline{a+x}\times\overline{\frac{b^2a^2}{b^2-c^2}+\frac{2ab^2+2bc^2}{b^2-c^2}\times x+x^2}|^{\frac{n+1}{2}}}{n+1}$$

$-\frac{\dot{x}\times\overline{a+x}|^{n+2}}{n+1}=$ the Fluxion of the Attraction,

which, by putting $z=x+\frac{ab^2+bc^2}{b^2-c^2}$, will become

$$\frac{\overline{b^2-c^2}|^{\frac{n+1}{2}}}{\overline{n+1}\times b^{n+1}}\times z\dot{z}\times\overline{z^2-\frac{b^2c^2\times\overline{c^2+2ab+a^2}}{\overline{b^2-c^2}|^2}}|^{\frac{n+1}{2}}-\frac{c^2\times\overline{a+b}}{b^2-c^2}$$

$$\frac{\overline{\dot{z}\times z^2 - \frac{b^2c^2\times\overline{c^2+2ab+a^2}}{\overline{b^2-c^2}|^2}}\Big|^{\frac{n+1}{2}} - \overline{\dot{z}\times z - \frac{c^2\times \overline{a+b}}{b-c^2}}\Big|^{n+2}}{n+1}$$

where, it appears that, the Fluents of the first and last Terms may be easily found; though that of the second cannot be had in finite Terms, unless in some particular Cases; as when $n=1$, 3, 5, 7, &c. respectively. Neither can it be found, but under the same Restrictions, when a is $=0$, in which Case the Expression does become $=\frac{x\dot{x}\times\overline{2px\pm dx^2}|^{\frac{n+1}{2}}}{b^{n+1}\times\overline{n+1}} - \frac{x^{n+2}\dot{x}}{n+1}$ where $2p=2bc^2$, and $\pm d=b^2-c^2$, according as b is greater or less than c.

Corollary. If $n=-2$, the Expression of the Fluxion at the Surface in one of the Poles will become $\dot{x}-\frac{bx\dot{x}}{\sqrt{2px\pm dx^2}}$; whose Fluent, when b is less than c, is $x-\frac{b}{\sqrt{d}}$ into the Difference between the Arch and the Sine of the Circle whose versed Sine is x and Radius $\frac{p}{d}$; which, when $x=2b$, will be $2b-\frac{b}{\sqrt{d}}$ into the Difference between the Arch and the Sine of the Circle whose versed Sine is $2b$ and Radius $\frac{p}{d}$: But if b is greater than c, the Fluent will be $x-\frac{b}{\sqrt{d}}$ into $\sqrt{\frac{2px}{d}+x^2}-\frac{p}{d}\times$Hyperbolic Logarithm

$p+$

$\frac{p+dx+d\sqrt{\frac{2p}{d}x+x^2}}{p}$; which, when $x=2b$, will be

$$2b-\frac{bp}{d^{\frac{3}{2}}}+\text{Hyp. Log.}\frac{p+2pd+d\sqrt{\frac{4pd}{d}+b^2}}{p}:\frac{b}{d^{\frac{1}{2}}}$$

$$\times\sqrt{\frac{4pb}{d}+4b^2}=2b+\frac{b^2c^2}{\overline{b^2-c^2}\big|^{\frac{3}{2}}}+\text{Hyp. Log.}$$

$\frac{2b^2-c^2+2b\sqrt{b^2-c^2}}{c^2}:-\frac{2b^3}{b^2-c^2}$, the Attraction of the whole Speroid.

A COL-

A COLLECTION OF PROBLEMS

To be anſwered in the next NUMBER.

PROBLEM LXXVII. *by Mr.* Hammond *of the* Bank.

THE Baſe of a plain Triangle being equal to 80, the vertical Angle equal to 84 Degrees, and the Sum of the Perpendicular and the leſſer Segment of the Baſe equal to 68; to determine the Triangle.

PROBLEM LXXVIII. *by Mr.* Moſs.

To deſcribe a Circle, through two given Points, ſo that the Segment of the Baſe cut off by an indefinite right Line, given in Poſition, ſhall contain a given Angle.

PROBLEM LXXIX. *by Mr*: Moſs.

The three Sides of a right-angled Triangle being equal to 100, 80 and 60; to draw two Lines, one of them from the right Angle to the Hypothenuſe and the other from the Point where the former meets the Hypothenuſe to either of the Legs, ſo as to divide the Triangle into three Parts, whereof that included by the ſaid Lines ſhall be a Geometrical Mean between the other two.

PROBLEM LXXX. *by* John Turner.

What is the Declination of the Plane, upon which the Sun, on *June* the 10 in the Latitude of 51°.32′, continues 9 Hours 50 Minutes?

PROBLEM LXXXI. *by* John Turner.

The Latitude of the Place being given; to determine the Azimuth Circle in which the Difference of the Altitudes of the Sun, in any two proposed Days of the Summer Half-year, shall be a Maximum.

PROBLEM LXXXII. *by Mr.* S. Ashby *of* London.

A Gentleman desires to have a Mount raised in his Park, in Form of the Frustrum of a Cone, which should be formed by the Earth to be thrown out of a Ditch to surround the same: Now, supposing that the Altitude of the Mount, above the Level of the Horizon, is 50 Feet, the Diameter of the uppermost Base 40 Feet, and that of the undermost 100 Feet; also, that the Breadth of the Ditch at its Bottom is 10 Feet, the innermost Side coincides with the slant Side of the Frustrum produced, and that the outermost Side makes an Angle of 135° with the Plane of the Bottom; 'tis required to determine the Altitude of the uppermost Base above the Bottom of the Ditch, and the Breadth of the Ditch in the Plane of the Horizon.

PROBLEM LXXXIII. *by* John Turner.

Supposing two unequal Weights, one of four and the other of two Pounds, to be suspended upon a Pin, by Means of a String; to determine how far the greater will descend and the lesser ascend in one Second of Time, neglecting the Friction of the Pin.

PROBLEM LXXXIV. *by Mr.* S. Clark *of* London.

To determine that Curve in which, supposing the Abscissa to be taken from the Subtangent, the Remainder shall be to the Abscissa, as the Tangent to the Ordinate.

PROBLEM LXXXV. *by* John Turner.

To determine the Equation of a Curve, by whose Revolution a Solid is generated, which at all Altitudes is exactly three Tenths of its circumscribing Cylinder.

PROBLEM

Problem LXXXVI. *by* John Turner.

To find the Equation of a Curve, to which a Perpendicular, to any Point in the Periphery of a given Ellipsis, shall always be a Tangent.

Problem LXXXVII. *by Mr.* Thomas Perryam.

To determine the Value of x, when $\left(x^{\frac{1}{x}}\right)^{x^{\frac{1}{x}}}$ is either a Maximum or a Minimum.

Problem LXXXVIII. *by* John Turner.

The Random of a Piece on the Plane of the Horizon, with a given Charge of Powder, at an Elevation of 30 Degrees being 1500 Yards; to find the Elevation of the same Piece, when planted at 44 Yards above the Level of the Horizon, so that (the Charge of Powder continuing the same) the Ball may fall at the greatest Distance possible; which Distance is also required.

Problem LXXXIX. *by* John Turner.

Of all the Parabolic Conoids that can be inscribed in a given Cone, to determine that whose Curve Surface will be the greatest possible.

Problem XC. *by* John Turner.

Suppose a Ball to be projected with a Velocity of 400 Feet in a Second, at an Elevation of 45°, and that the Resistanee of the Atmosphere, at its leaving the Mouth of the Piece, is to the Force of Gravity in a given Ratio (suppose that of Equality); to determine the Distance the Ball must move before it arrives at its greatest Altitude, its Velocity answering to the said Altitude, together with the Distance described by the Ball when its Velocity is a Minimum; supposing the Law of Resistance to be as the Square of the Celerity.

The End of NUMBER V.

THE

Mathematician.

DISSERTATION VI.

Upon the Progress and Improvement of GEOMETRY.

OUR two last Dissertations contain a large Account of the Nature and Definition of Fluxions, and the manner of expressing the Relations thereof, not only in Magnitudes purely geometrical, but also shew that the same was extensible to all other kind of algebraical, logarithmical, or exponential Quantities, which vary by Increase or Decrease.

We proceed now to the third and last Part of our Design, *viz.* to shew the Application and Use of this Method: Its great Author comprizes the whole

of it in the Solution of two general Problems belonging to the abstract, or rational Mechanics; for the direct Method of Fluxions, as it is now called, amounts to this mechanical Problem, *The Length of the Space described being continually given, to find the Velocity of the Motion at any Time proposed.* Also the inverse Method of Fluxions has for its Foundation, the Reverse of this Problem, which is, *The Velocity of the Motion being continually given, to find the Space described at any Time proposed.* So that upon the compleat analytical or geometrical Solution of these two Problems, in all their Varieties, he builds his whole Method.

His first Problem, which is in his Quadratures of Curves, *viz. An Equation being given, involving any Number of flowing Quantities, to determine the Relation of their Fluxions*, he dispatches very generally. He does not propose this, as is usually done, *a flowing Quantity being given, to find its Fluxion*; for this gives us too lax and vague an Idea of the Thing, and does not sufficiently shew that Comparison, which is here always to be understood. Fluents and Fluxions are things of a relative Nature, and suppose two at least concerned, whose Relation or Relations should always be expressed by Equations. He requires therefore, that all should be reduced to Equations, by which the Relation of the flowing Quantities will be exhibited, and their comparative Magnitudes will be more easily estimated.

To fix the Ideas of his Readers, he illustrates his general Problems, by a particular Example. If two Spaces x and y are described by two Points, in such manner, that the Space x being uniformly increased, in the Nature of Time, and its equable Velocity being represented by the Symbol $\dot{x}$; and if the Space y increases inequably, but after such a Rate, as that the Equation $y=xx$ shall always determine the Relation between those Spaces, (or x being continually

given, as having any certain Value, y will thence be likewise known, by the Rules of common Algebra) then the Velocity of the Increase of y shall always be represented by $2x\dot{x}$. That is, if the Symbol $\dot{y}$ represent the Velocity of the Increase of y, then will the Equation $\dot{y}=2x\dot{x}$ always obtain. For the Fluxions of equal flowing Quantities are equal, and by Differtation 4th, p. 202. the Fluxion of xx is $2x\dot{x}$.

Now from the given Equation $y=xx$, or, which is the same thing, from the relation of the Spaces y and x, (*i. e.* the Space and Time, or its Representative for Time, may be represented by a Space equably generated) being continually given, the Relation of the Velocities $\dot{y}=2x\dot{x}$ is found, or the Relation of the Velocity $\dot{y}$, by which the Space increases to the Velocity $\dot{x}$, by which the Representative of the Time increases. And this is an Instance of the Solution of the first general Problem, of which, however, we will more particularly explain in the Operation, by giving the Author's Rule, and illustrate it by some Examples.

RULE.

Let each Term of the Equation be multiplied by the Exponent of the Power of every flowing Quantity that Term includes, and in the several Multiplications, instead of the Root of any Power of each, substitute its Fluxion; then will the Sum of all those Products under their proper Signs, be the new Equation sought.

EXAMPLE I.

Let $x^3-xyy+aaz-b^3=0$ be the fluential Equation proposed, where a and b are standing Quantities; and x, y, z, flowing ones. Since x is in only two Terms of the Equation, multiply those Terms by the respective Exponents of the Powers thereof,

 which

which are 3 and 1, and we shall have $3x^3 - xyy$; and putting in $\dot{x}$ Fluxion, instead of the Root of those Powers of x, (that is, instead of $\dot{x}$ itself, or x of one Dimension) we shall then have $3x^2\dot{x} - \dot{x}yy$. If we proceed after the same manner with the quantity y, the Result of that will be $-2xy\dot{y}$; and if with the Quantity z, we shall have $aa\dot{z}$. So that the Sum of all the Products, with their respective Signs, is $3x^2\dot{x} - yy\dot{x} - 2xy\dot{y} + aa\dot{z}$. Put this Expression $=o$, then will their Equation give us the relation of the Fluxions of the flowing Quantities, (as so combined and related in the Equation proposed) or in other Words, this Equation will be the fluxional one of the other; which may be demonstrated from the Doctrine of Moments, explained in our last, thus: If at the present Instant, the flowing Quantities are z, y, x, at the next Instant, (when augmented by their respective Moments) they will become $z+o\dot{z}$, $y+o\dot{y}$, $x+o\dot{x}$. Substitute these Expressions, (in the Equation proposed) in the room of z, y, and x respectively; and then that Equation, which will still be a good one *, will

* If the Truth of this should be doubted, let the Terms be expanded and reduced to 3 Orders or Columns, according as the vanishing Quantity o is of none, one, or more Dimensions; thus.

Here the Terms of the first Order or Column remove or destroy one another, as being absolutely equal to nothing by the given Equation.

$$\left\{\begin{array}{lll} x^3 & +3x^2o\dot{x} & +3xo^2\dot{x}^2 \\ -xy^2 & -o\dot{x}y^2 & +o^3\dot{x}^3 \\ +a^2z & -2oxy\dot{y} & -2\dot{x}o^2y\dot{y} \\ -b^3 & +a^2o\dot{z} & -xo^2\dot{y} \\ & & -\dot{x}o^3\dot{y}^2 \end{array}\right\} = o$$

They being therefore expunged, the remaining Terms may be all divided by the common Multiplier o, whatever it is. This being done, all the Terms of the third Order will still be affected by o, of one or more Dimensions, and may therefore be expunged, as infinitely less than the others. Lastly, there will only remain those of the second Order, or Column $3x^2\dot{x} - y^2\dot{x} - 2xy\dot{y} + a^2\dot{z}$, which will be the fluxional Equation required.

appear

appear thus, *viz.* $x^3+3x^2o\dot{x}+3xo^2\dot{x}^2+o^3\dot{x}^3-xy^2-o\dot{x}y^2-2xoy\dot{y}-2\dot{x}o^2y\dot{y}-xo^2\dot{y}^2-\dot{x}o^3\dot{y}^2+a^2z+a^2o\dot{z}-bz=o$.

The Equation propofed being comprehended in this, fubftract that from this, and divide the Remainder by the Quantity o, and then it will be, $3x^2\dot{x}+3xo\dot{x}^2+o^2\dot{x}^3-y^2\dot{x}-2xy\dot{y}-2oy\dot{x}\dot{y}-xo\dot{y}^2-o^2\dot{x}\dot{y}^2+aa\dot{z}=o$. Now let the Quantity o be diminifhed infinitely, or fuppofed to vanifh, and then all the Terms into which it is multiplied, alfo vanifhing, there will remain $3x^2\dot{x}-y^2\dot{x}-2xy\dot{y}+aa\dot{z}=o$, the fluential Equation, which determines the Relation of the Fluxions, as far as it can be determined from one Equation, only including the flowing Quantities.

This Procefs is in effect the fame with that which has been fo often mentioned before, for determining the Proportions of Fluxions; *viz. the laft Ratio of the Increments:* To make it as plain as poffible, Mr. *Ditton* works an Equation according to both Methods, and compares the Refults. Let the Equation be $aaz-xyy=o$, or $aaz=xyy$. Then according to the Rule above, the fluxional Equation will be $aa\dot{z}-yy\dot{x}-2y\dot{y}x=o$. Now, if by the other Method, we take the laft Ratio of the Increments generated in a given Particle of Time, that will give us the Proportion and Relation of the Fluxions of any fluent Quantities propofed. To ftate the Increments rightly and congruoufly, we are to confider, that in the fame Inftant of Time the Quantities x, y, and z, flow into $x+o\dot{x}$, $y+o\dot{y}$, and $z+o\dot{z}$; and in the fame Inftant that y becomes $y+o\dot{y}$, yy becomes $yy+2oy\dot{y}+oo\dot{y}^2$; and in the fame Inftant that yy becomes $yy+2oy\dot{y}+oo\dot{y}^2$, the Quantity xyy becomes $xyy+2xyo\dot{y}+xoo\dot{y}^2+oyy\dot{x}+2yoo\dot{y}\dot{x}+o^3\dot{x}\dot{y}^2$; and the Quantity aaz becomes $aa\dot{z}+aao\dot{z}$. Therefore the Augments of xyy and aaz generated in the fame Inftant, are $2xyo\dot{y}+xoo\dot{y}^2+oy^2\dot{x}+2yoo\dot{y}\dot{x}+o^3\dot{x}\dot{y}^2$, and $aao\dot{z}$; and thefe are evidently one to another, as $2xy\dot{y}+xo\dot{y}^2+yy\dot{x}+2yo\dot{y}\dot{x}+o^2\dot{x}\dot{y}^2$

$+o^2\dot{x}\dot{y}^2$ to $aa\dot{z}$. And for the laſt Ratio of them let the augmenting Quantity o vaniſh; and then they will be as $2xy\dot{y}+yy\dot{x}$ to $aa\dot{z}$, which is the Relation of the Fluxions required. But now, becauſe the flowing Quantities are equal, $xyy=aaz$, therefore their Increments generated in a given Particle of Time are equal; *i. e.* the Ratio of thoſe Increments conſidered as *finite*, is a Ratio of Equality; and therefore the Ratio of thoſe Increments conſidered as *naſcentia*, or *evaneſcentia*, ſhall be a Ratio of Equality; and therefore the Ratio of the Fluxions ſhall be a Ratio of Equality; that is, $2xy\dot{y}+yy\dot{x}=aa\dot{z}$, and conſequently $aa\dot{z}-yy\dot{x}-2xy\dot{y}=o$, which is the Equation found by the Rule above. The Meaning of all this muſt be, that whenever $\dot{x}$ and $\dot{y}$ denote the Velocity of flowing of x and y; then the fluxional Equation will expreſs the Velocity wherewith the whole compounded Fluents in the fluential Equation flows.

In this way of arguing, there is no Aſſumption made, but what is juſtifiable by the received Methods, both of the antient and modern Geometricians. We only deſcend from a general Propoſition, which is undeniable, to a particular Caſe, which is certainly included in it. That is, having the Relation of the variable Quantities, we thence directly deduce the Relation or Ratio of their contemporary Augments; and having this, we directly deduce the Relation or Ratio of thoſe contemporary Augments, when they are naſcent or evaneſcent, juſt beginning, or juſt ceaſing to be; in a Word, when they are Moments, or vaniſhing Quantities.

To evade this Reaſoning, it ought to be proved, that no Quantities can be conceived leſs than aſſignable Quantities; that the Mind has not the Privilege of conceiving Quantity as perpetually diminiſhing *ſine fine*; that the Conception of a vaniſhing Quantity, a Moment, an infiniteſimal, &c. includes a Contradiction. In ſhort, that Quantity is not (even mentally)

tally) divisible *ad infinitum*; for to that the Controversy must be reduced at last. But it will be a very difficult Matter to extort this Principle from the Mathematicians of our Days, who have been so long in quiet Possession of it, who are indubitably convinced of the Evidence and Certainty of it, who continually and successfully apply it, and who are ready to acknowledge the Fertility and extreme *Usefulness* of it, upon so many important Occasions.

In order to ascertain and express the Relation of these Fluxions (concerning which we have said so much) in Numbers, let us take another Example. The fluential Equation is $zy^3 - z^4 + a^4 = 0$, the fluxionary Equation of which, according to the above Rule, will be $3zy^2\dot{y} + \dot{z}y^3 - 4z^3\dot{z} = 0$; this reduced to an Analogy shews the Relation or Ratio of $\dot{z}$ to $\dot{y}$ to be $\frac{\dot{z}}{\dot{y}} = \frac{3zy^2}{4z^3 - y^3}$; the first of these Equations expresses the Relation of the flowing Quantities z and y at all Times, or in every State; and the second shews the Relation of their Fluxions, at all Times, in Terms made up of z and y. Wherefore, if we assume a particular determinate Value for z or y, the corresponding Value of the other may be found, from the first Equation; and thereby the Ratio of $\dot{z}$ and $\dot{y}$, or $\frac{\dot{z}}{\dot{y}}$, (*viz.* the Rate of flowing, or Ratio of the Fluxions at different Values of the Fluents) will be wholly known.

For Instance, suppose that any Time $z = 2a$; then by substituting $2a$ for z, in the Equation $zy^3 - z^4 + a^4 = 0$, we have $2ay^3 - 15a^4 = 0$, whence $y^3 = \frac{15a^4}{2a} = \frac{15}{2}a^3$, or $y = a^3\sqrt{\frac{15}{2}}$. Substitute these Values of z and y, in the Expression of the Ratio of $\dot{z}$ and $\dot{y}$, and

it becomes $\frac{\dot{z}}{\dot{y}} = \frac{\frac{6a^3\sqrt[3]{225}}{4}}{32a^3 - \frac{15a^3}{2}} = \frac{\frac{12\sqrt[3]{225}}{4}}{49} = \frac{\sqrt[3]{97200}}{\sqrt[3]{117649}}$

wholly a known Quantity. Which ſhews, that at what time or place the variable Quantity z becomes $=$ the known determinate Quantity $2a$, the Velocity with which z flows at that Time, is to the Velocity with which y flows at the ſame Time, as the cube Roots of 97200, and 117649, are to one another. And if z be taken of any other Value, another known Relation of $\dot{z}$ and $\dot{y}$ will ariſe.

Hence it appears, that when a fluential Equation is propoſed, containing two variable or flowing Quantities only, the Relation of the firſt Fluxions of theſe two flowing Quantities, may always be had in Terms containing the two variable Quantities, and known Quantities only; and therefore by aſſuming one of the flowing Quantities at Pleaſure, the fluential Equation by the Rules of common Algebra, will give a correſponding known Value of the other flowing Quantity: Wherefore the Relation which the Fluxions of theſe two Quantities bear to one another at that Time, will be fully known and determined, by ſubſtituting the particular determined Values of the flowing Quantities, in place of them in the fluxionary Equation. And one of the Fluxions, (whoſe Fluent flows uniformly) being taken for Unity, or of any determinate Value, the Value of the other may be exhibited by a Number, which will be a compleat Determination.

For another Example, let the fluential Equation $px - y^2 = o$ be propoſed, which belongs to the common Parabola; x being the Abſciſs; y, the Ordinate; and p, the Parameter of the Diameter. The fluxionary Equation thence ariſing, is $p\dot{x} - 2y\dot{y} = o$; nce $\dot{x} : \dot{y} :: 2y : p$; *viz.* the variable Ratio of $\dot{x}$

and $\dot{y}$, will be at all Times, and in every Place, as twice the Ordinate in that Place to the Parameter. Now suppose we would know, what that Ratio would be, when $x = p$, insert p for x in the Equation $px - y^2 = o$, and it becomes $p^2 - y^2 = o$, or $p = y$; so that when the Absciss is $=$ the Parameter, so is the Ordinate likewise. Wherefore insert p for y, in the Value of the Ratio of $\dot{x}$ to $\dot{y}$, and it is $\dot{x} : \dot{y} :: 2p : p :: 2 : 1$: So that, at that Time, the Velocity with which the Absciss encreases, is double the Velocity with which the Ordinate increases. And so by assuming other Values of x, or of y, other known Relations of $\dot{x}$ and $\dot{y}$ will arise.

But if there are three variable or flowing Quantities in a fluential Equation proposed, another Equation including at least two of them, ought also to be given, that the Relation of their Fluxions may be fully determined; and also their Relation among themselves. Thus let the fluential Equation $ax + by^2 - cxz = o$ be given, including three flowing Quantities, x, y, and z. The fluxional Equation thence deduced is $a\dot{x} + 2by\dot{y} - cx\dot{z} - cz\dot{x} = o$, which gives the Relation of the Fluxions $\dot{x}$, $\dot{y}$, and $\dot{z}$, as far as that Relation can be determined from one fluential Equation only. But the Relation of the Fluents in the fluential Equation, and of their Fluxions in the fluxionary one, will not be fully determined, unless another fluential Equation be given. As if it were supposed, that $x - ay + z = o$. From whence we deduce $\dot{x} - a\dot{y} + \dot{z} = o$, for another Relation of the Fluxions, besides the former. Therefore by comparing the two fluential Equations together; and the two fluxional Equations thence deduced together, we may exterminate any one of the flowing Quantities, and also any one of the Fluxions; and thereby we may obtain an Equation, which will entirely determine the Relation of the other two, whether flowing Quantities or Fluxions. Thus, if you want to have

the Relation of $\dot{x}$ and $\dot{y}$ fully determined, the first fluxional Equation gives $\dot{x} = \frac{a\dot{x} + 2by\dot{y} - cx\dot{z}}{cx}$, the other, which is deduced from the second fluential Equation, &c. either given, or assumed, will give $\dot{z} = a\dot{y} - \dot{x}$; therefore by equating these Values of $\dot{z}$, we have $\frac{a\dot{x} + 2by\dot{y} - cx\dot{z}}{cx} = a\dot{y} - \dot{x}$. Again, by means of one of the fluential Equations, as of $x - ay + z = o$, take a Value of z; *viz.* $z = ay - x$, and put that in Place of z in the Equation $\frac{a\dot{x} + 2by\dot{y} = c\dot{x}z}{cx} = a\dot{y} - \dot{x}$, and it becomes $a\dot{x} + 2by\dot{y} - cay\dot{x} - cax\dot{y} + 2cx\dot{x} = o$, or $\dot{x} : \dot{y} :: acx - 2by : a - cay + 2cx$. Where, if you take x at pleasure, the fluential Equations will give y, and so the Relation of $\dot{x}$ and $\dot{y}$ will be entirely determined. Or more shortly thus. From the fluential Equation $ax + by^2 - cxz = o$ proposed; and the other $x - ay + z = o$, either given or assumed, find another Equation free of z, which will be $ax + by^2 = acxy - cx^2$; and from that you'll have $a\dot{x} + 2by\dot{y} - ac\dot{x}y - acx\dot{y} - 2cx\dot{x}$, as formerly.

We may observe, that when there are three flowing Quantities, and but one Equation to determine their Relation by, the Fluxions, as well as the Quantities themselves, admit of an infinite Variety of Relations, (like indeterminate Problems in common Algebra, which admit of various Answers) in that Case we must assume another Equation, as above, whereby to determine one of these Relations only; for there ought always to be given as many Equations save one, as there are flowing Quantities. And further, the Fluxions of homogeneous Quantities are still to be considered in relation to one another; for without such a Consideration, we can make nothing of the Doctrine of Fluxions. Therefore, since every fluxionary Equation contains the Fluxions of two flowing

flowing Quantities at least, either expressed or understood; and thereby determines only the Relation of these Fluxions; it is left at Liberty, to suppose one of these flowing Quantities to flow or change at any rate; either equably and uniformly; or according to any Law of Acceleration or Retardation, we think fit to frame or suppose: because such a Supposition never alters the Relations of the Fluxions, or Velocities of flowing. Hence our Conception of the Relation of Fluxions, will then be most clear and distinct, when we suppose one of the flowing Quantities to flow with an uniform and invariable Velocity, and call it Unity. For thereby that uniform Fluxion is made a common Standard of Magnitude, by which to measure the rest.

Thus in the Equation of a common Parabola $px-y^2=0$, from which the fluxionary Equation $p\dot{x}-2y\dot{y}=0$ is deduced: we suppose the Abscis x to flow uniformly, and put its Fluxion $\dot{x}=1$; by this means, the fluxional Equation becomes $p-2y\dot{y}=0$, i. e. $\dot{y}=\frac{p}{2y}$; so that the Fluxion of the Abscis being always 1, the Fluxion of the Ordinate will always be expressed by the Quantity $\frac{p}{2y}$. We might have supposed the Parabola to be so described, by the Motion of the Ordinate along the Axis, that the Velocity with which the Ordinate increases, is always invariably the same, and call it Unity; then $p\dot{x}-2y\dot{y}=0$, becomes $p\dot{x}-2y=0$ or $\dot{x}=\frac{2y}{p}$, i. e. in such a Case, the Velocity with which the Abscis flows will be always expressed by $\frac{2y}{p}$. And so of others.

The Application of this first general Problem is extensive, for by it we are enabled to solve the following useful Problems; *viz.*

1. To determine the *Maxima* and *Minima* of Quantities.

2. To draw Tangents to all Sorts of Curves, whether geometrical or mechanical.

3. To determine the Points of contrary Flexure, and Retrogression in Curves.

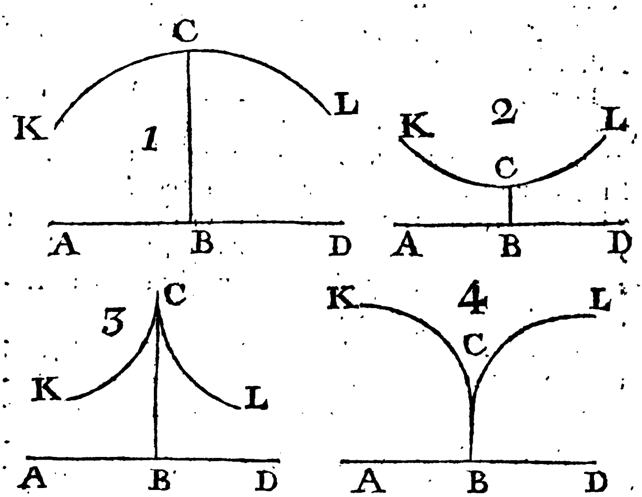
4. To determine the Quantity of Curvature at a given Point of a given Curve; *i.e.* to find the Length of the Radius of an equi-curve Circle.

Under this may be comprehended the finding the Points, where a Curve has any given Degree of Curvature; where it has the greatest or least, and the Locus of the Centre of Curvature.

To exemplify what has been said in an Instance or two, of the Doctrine of *Maxima* and *Minima*, (tho' Mr. *Fermat* a Century ago, and Mr. *Simpson*, in his Elements of plain Geometry, has solved several Problems of this Kind, yet the Method of Fluxions comprehends and improves all others, especially the most difficult) for the famous Rule of *Huddenius*, which multiplies the Equation, when properly disposed by an arithmetical Progression, extends not to Equations affected with surd Quantities. Some of these Problems relate to the Ordinates of Curves; for when an Arch of a Curve has its Concavity turned only one way, and there is a Point in this Arch, where the Tangent becomes parallel or perpendicular to the Abscis, then the greatest or least Ordinate passes thro' this Point: the Writers upon the practical Part of this Doctrine commonly distinguish these, by supposing the Abscis to increase, and observing whether the Fluxion of the Ordinate be positive, before the Abscis arrives at another certain Point, and becomes negative afterwards; or is first negative, and then becomes positive; it is a *Maximum* in the former, and a *Minimum* in the latter Case. The Rules they give for discovering these, direct us to find an Equation according to the Pro-

perties of the Figure, expressing the Quantity sought, in Terms of the Abscifs, Ordinate and known Quantities; then we are bid to *throw the whole Expression into Fluxions*, striking out every Term which is multiplied into the Fluxion of the Ordinate, and from the remaining Terms, when made $= o$, a Value of the Abscifs will be found, when the Ordinate is a *Maximum* or *Minimum*.

If it should here be objected by any, that these Directions seem not sufficiently scientific, nor appear connected with the Nature of Fluxions, as hitherto explained, and should demand *why* and *wherefore* he should do as above directed: To shew that true Mathematics require nothing to be effected as it were by *Charms*, or *hocus pocus Tricks*, we will endeavour to give the Rationale of this Matter thus; let KCL be a Curve described, with the Base AD, and Ordinate BC, on both Sides of which the Curve extends:



While the Abscifs AB encreases, as at Fig. 1 and 3, the Ordinate BC encreases together with it, until it arrive at the Position represented in these Figures, afterwards

afterwards it decreases; therefore at that Instant of Time when BC is between increasing and decreasing, it attains a greater Length, than in the Times or Positions a little preceding or succeeding, and therefore is called a *Maximum*: thus circumstanced, 'tis a standing Quantity, having no Velocity of Increase or Decrease, consequently no Fluxion, or its Fluxion is $=o$, and therefore whatever it is multiplied with is $=o$, as at Fig. 1. or else is infinitely great in comparison of the Fluxion of AB, as at Fig. 3. otherwise from being positive, it could not become negative.

Again, at Fig. 2 and 4, while AB increases, BC diminishes, until it arrive at the Position represented in these Figures; therefore in the intermediate Instant 'tis called a *Minimum*, when the Fluxion of BC, from having been negative, next becomes positive; consequently at the exact Instant, for the Reasons above, must either be $=o$, as at Fig. 2. or be infinitely great, in comparison of the Fluxion of AB, as at Fig. 4.

Hence it is evident, that we call x the Absciss, and y the Ordinate of a Curve, when the Fluxion of y vanishes, the Tangent becomes parallel to the Absciss, as in Fig. 1 and 3; and when the Fluxion of y is infinitely great, in comparison of the Fluxion of x, the Tangent coincides with the Ordinate, as in Fig. 2 and 4. (Points of contrary Flexure and Retrogression are not here considered.)

Wherefore an Equation being given, including two unknown Quantities x and y, which may be conceived as the Absciss and Ordinate of a Curve, the last of which is to be determined at its extreme Value; find the Relation of their Fluxions, or fluxionary Equation, as before shewn; (vulgarly called throwing the Expression into Fluxions) then put $\dot{y}$ or $\frac{\dot{y}}{\dot{x}}=o$,

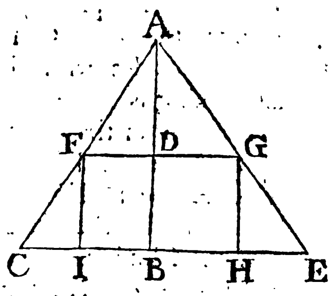
because

because by the Nature of the Figure, and for the Reasons above, it must be so; from the Equation thence resulting, compared with the fluential Equation, we may exterminate the unknown Quantity, and get a new Equation, including only the other, x, whose Root or Roots will shew the Point or Points of the Abscis, at which an Ordinate or Ordinates being applied, will pass thro' a Point or Points of the Curve, where the Tangent is parallel to the Abscis; and again, by putting $\frac{\dot{x}}{\dot{y}}=o$, (*i.e.* changing the Ratio of the Fluxions) and proceeding the same way, we may discover the Points of the Curve, where the Tangent coincides with the Ordinate. Sometimes the same thing is done by putting the Reciprocal of the Fluxion of the Quantity, whose extreme Value is to be determined $=o$.

Let $y+x^2-2ax-b^2=o$ be an Equation proposed, and it is required to determine the extreme Value of y. The Relation of the Fluxions, or the fluxionary Equation, is $\dot{y}+2x\dot{x}-2a\dot{x}=o$. Now since $\dot{y}$ is necessarily $=o$, therefore $\frac{\dot{y}}{\dot{x}}=o=2a-2x$, hence we have $x=a$. To find whether y be at an extreme Value; and if so, whether it be a *Maximum* or *Minimum*, put $x+p$, and $x-p$ instead of x, in the Equation $y=b^2+2ax-x^2$; whence we have first, $y=b$ $+2ax+2ap-x^2-2px-p^2$. And second, $y=b^2+2ax-2ap-x^2+2px-p^2$. From which two last Values of y substract $b^2+2ax-x^2$, and the Differences are, first, $+2ap-2px-p^2$; and second, $-2ap+2px-p^2$, where if for x, its corresponding Value a be inserted, and the repugnant Terms thrown out, the resulting Differences are on both Sides the Ordinate $-p^2$, therefore y is a *Maximum*, since both Differences are negative: had they both been positive, it would have been a *Minimum*; if one had been positive,

positive, and the other negative, there would have been a Point of contrary Flexure.

Again, if we put $\dot{y}$ infinite in respect of $\dot{x}$, or $\frac{\dot{x}}{\dot{y}}=o$, then we have $\frac{1}{2a-2x}=\left(\frac{\dot{x}}{\dot{y}}\right)=o$; which can only happen when x is infinitely great, where no *Maximum* or *Minimum* can take place; for while the Divisor has any assignable Magnitude, the Quotient will be something.



Let it be required to inscribe in any given Triangle ACE, the greatest Parallelogram possible FH.

Call AB$=a$, CE$=b$, AD$=x$, therefore DB$=a-x$; per similar Triangles, $a:b::x:\frac{bx}{a}=$FG; but FG$\times$DB, *i. e.* $\frac{bx}{a}\times a-x=\frac{abx-bx^2}{a}$, is a *Maximum*; or (because $\frac{b}{a}$ is a standing Quantity) $ax-x^2$ is a *Maximum*: wherefore suppose $ax-x^2=y$. where you may imagine x and y to be related, as the Absciss and Ordinate of a Curve, in which you are to determine y to an extreme Value. Hence you'll have $\frac{\dot{y}}{\dot{x}}=o=a-2x$, whence arises $x=\frac{1}{2}a$. *i. e.* AD$=$AB, in order to make the inscribed Parallelogram a *Maximum*.

After

After the ſame manner you might determine the inſcribed Parallelogram to a *Maximum*, if CAE had been a Segment of any known Curve: thus, if it had been a Portion of a Parabola having AB for the Diameter, and CE an Ordinate bounding the Fig. then becauſe per Conics $a : x :: b^2 : \frac{b^2 x}{a} = FG = b\sqrt{\frac{x}{a}}$; wherefore ſuppoſe this Expreſſion, when multiplied by $a - x$ (becauſe FG×DB is the Parallelogram required) $= y$, take the Relation of the Fluxions, or fluxionary Equation, and for the Reaſons above make $\frac{\dot{y}}{\dot{x}} = o$, then we have the Fluxion of $\overline{a-x} \times b\sqrt{\frac{x}{a}}$, or, which is the ſame thing, (by dividing by the given Quantity $\frac{b}{\sqrt{a}}$) of $ax^{\frac{1}{2}} - x^{\frac{3}{2}} =$ nothing; *i. e.* $\frac{1}{2}ax^{-\frac{1}{2}}\dot{x} - \frac{3}{2}x^{\frac{1}{2}}\dot{x} = o$, whence by reducing you find $x = \frac{a}{3}$, which is true from other Principles. Under this Branch are comprized great Variety of curious and important Problems in Mechanics, Aſtronomy, and Phyſics.

We ſhall now ſhew how the great Inventor applied his firſt general Problem above, to the *drawing of Tangents to all manner of Curves*, not to geometrical ones only, where *Sluſius*'s Method reaches, but this extends even to mechanical Curves alſo: And the manner of Operation herein eaſily follows, from what was before proved in Diſſertation 5th, p. 266. concerning the Relation of the Fluxions of the Abſciſs and Ordinate. For ſince it appears, by calling the Abſciſs x, the Ordinate y, and the Subtangent s, that $\dot{y} : \dot{x}$: : Ordinate : Subtangent. Hence the general Formula or Expreſſion for the Subtangent is $\frac{\dot{x}}{\dot{y}}y = s$; if therefore from the Equation of

the Curve, we find the fluxionary Equation; and if, instead of the Ratio of the Fluxions in the Formula, we substitute the Ratio equal to it, which results from the fluxionary Equation, and is composed of other Quantities, freed of Fluxions; we shall then have an Expression for s, in Terms of the Ordinate, and other known Terms.

Let $y^n = a^{n-1}x$ define the Nature of the Curve; this put in Fluxions, as it is commonly phrased, will be $ny^{n-1}\dot{y} = a^{n-1}\dot{x}$ or $\frac{\dot{x}}{\dot{y}} = \frac{ny^{n-1}}{a^{n-1}}$; multiply this by y, then instead of the general Formula above, we have $\frac{ny^n}{a^{n-1}} \left(= \frac{\dot{x}}{\dot{y}} y \right) = s$, the Length of the Subtangent; but if we would have the Subtangent expressed in Terms of the Abscifs, insert for y^n its Value $a^{n-1}x$, from the Equation of the Curve, and we have $nx = s$: wherefore if we take $n=2$, the Equation will belong to the common Parabola, and so the Subtangent will be twice the Abscifs. If n be 3, 4, 5, &c. which gives a Series of Parabola's of different Orders, we shall have the Subtangent equal to 3 times, 4 times, 5 times, &c. the Abscifs respectively.

Altho' the Curves proposed were mechanical; *viz.* the Cycloid, common Spiral, &c. wherein one of the flowing Quantities is a curve Line, and whose Relation to the other cannot be generally defined by an algebraic Equation, yet may Tangents be drawn to them; for all that is necessary is to find the Relation of the Fluxions of the Abscifs and Ordinate, expressed by finite Lines from the Property of the Curve, and substituting the proper Expressions thence resulting in the general Formula for the Subtangent above. The first general Problem, whose Use we have exemplified in the *Maximis* and *Minimis*, and in drawing Tangents to Curves, does likewise extend to other Branches, such as the finding Points of

of contrary Flexure; and the Evoluta of Curves, which have their Uſes in mathematical Philoſophy; and this by means of ſecond, third, and ſuperior Orders of Fluxions. Altho' Sir *Iſaac*, in his *Method of Fluxions and infinite Series*, purſues the above-mentioned Speculations, which require the Uſe of ſecond Fluxions, or higher Orders, and has there very artfully contrived to reduce them to firſt Fluxions, and to avoid the Neceſſity of introducing thoſe of ſuperior Orders. Yet in his other excellent Works of this kind, as he makes expreſs mention of them, we will attempt to diſcover their Nature and Properties. When a Fluent is not augmented or diminiſhed by an uniform, but by a varying Velocity, then its Fluxion or Celerity of flowing, being different at different times, is *itſelf* a variable, indeterminate, or flowing Quantity, and therefore admits of a Fluxion, called the Fluxion of the Fluxion, or ſecond Fluxion of the Fluent. If this ſecond Fluxion be conſtant and invariable, there is no third Fluxion; but if otherwiſe, that variable Fluxion or Celerity may be *itſelf* conſidered as a flowing Quantity, and therefore admits of a Fluxion, as well as any other variable Quantity; and is called the ſecond Fluxion of the firſt Fluxion, or the third Fluxion of the flowing Quantity; and ſo on without End. For the Fluxion of any flowing Quantity, being nothing elſe but its Celerity of flowing; and Celerity being itſelf a Quantity; there is no reaſon why, when it is variable, it ſhould not be conſidered in the ſame light with any other flowing Quantity; *i. e.* as having a Fluxion, which expreſſes the ſwifter or ſlower Mutation, with which that Celerity flows or changes.

In the two Fluents AR and CS, whoſe Fluxions we compared at p. 201. in Diſſertation 4, where AR being denoted by x, and CS by cx^{v}; the Fluxion of AR, bore to the Fluxion of CS, the Proportion of x to cvx^{v}, or of $1 : cvx^{v-1}$. Here it is evident, that

 the

the Antecedent of this Ratio being a fixed Quantity; *viz.* Unity and the Consequent cvx^{v-1}, while x flows, a variable one, unless when $v=1$; the Fluxion of AR does not bear to the Fluxion of CS always the same Ratio. If $v=2$, that Ratio is as 1, to the variable Quantity $2cx$; and if $v=3$, that Ratio is as 1 to $3cx^2$, still varying from the former Ratio. Therefore if AR be described with an uniform Velocity, when v is any Number greater than Unity, CS is so described with a Velocity continually accelerating, that when $v=2$, this Velocity augments in the same Ratio as AR itself increases, (for $2c$ a standing Quantity alters not the Ratio at all) and when $v=3$, it augments in the Duplicate of that Ratio, &c.

Here therefore we see, that while one Quantity flows uniformly, the other is described with a varying Motion; and the Variation in this Motion, is called the second Fluxion of this Quantity. It is farther evident, that in this Instance, when $v=2$, the Variation of the Velocity is uniform, for the Velocity keeping always the same Proportion as x; while x increases uniformly, the Velocity must also increase after the same manner. But when $v=3$, since the Velocity is every where as x^2, and x^2 does not increase uniformly, neither will the Velocity augment uniformly. So that it appears the Variation in the Velocity wherewith Magnitudes increase, may also vary; and these Variations are called Fluxions of Fluxions: this Consideration of Velocities thus perpetually varying, and their Variation *itself* changing is an useful Speculation, because it is true in fact, most bodies in nature we have any Acquaintance with, actually moving with Velocities thus modified; witness the true Theory of the Descent of Bodies, the Motion of the Planets in their elliptic Orbits, the Motion of Light at the Confines of different Mediums, and the Motion of all pendulous Bodies.

In

In short, an uniform unchangeable Velocity, is not to be met with in any of thoſe Bodies, that fall under our Cognizance: for in order to continue ſuch a Motion as this, it is neceſſary, that they ſhould not be diſturbed by any Force whatever, either of Impulſe or Reſiſtance; but we know of no Spaces, in which at leaſt one of theſe Cauſes of Variation does not operate. Wherefore the Velocity of a Velocity, how uncouth ſoever it may ſound to the Author of the Analyſt, will excite no abſurd Idea, when rightly conceived; but, on the contrary, will be a very rational and intelligible Notion; and conſequently, that of ſecond, third, and higher Orders of Fluxions, muſt be admitted as ſound and genuine: and every Order of them is aſſigned from the next lower, in the ſame manner as the firſt Fluxions are aſſigned from their Fluents.

We now proceed to the ſecond general and fundamental Problem of this Doctrine, which contains what is called the inverſe Method of Fluxions, and is borrowed from the Science of rational Mechanics; which is, from the Velocities of the Motion at all times given, to find the Quantities of the Spaces deſcribed; or to find the Fluents from the given Fluxions. It may be thus expreſſed, *An Equation being propoſed, involving the Fluxions of Quantities, to find the Relation of thoſe Quantities to one another.* This taken in its full Extent, is juſtly called *moleſtiſſimum & omnium difficillimum Problema.* The great Inventor firſt gives a particular, and then a general Solution of it. His particular Solution extends to ſuch Caſes only, wherein the fluxional Equation propoſed, either has been, or might have been derived from ſome finite algebraical Equation, which is now required. Here all the neceſſary Terms being preſent, and no more than what are neceſſary, we may by a Proceſs, juſt contrary to the former, return back again to the original Equation. Tho' to find the

the Fluent in finite Terms, when it can be done, requires particular Expedients. But it will most commonly happen, either if we assume a fluxional Equation at pleasure; or if we arrive at one, as the Result of some Calculation, that such an Equation is to be resolved, as could not be derived from any previous finite algebraical Equation, but will have Terms either redundant or deficient; and consequently the algebraical Equation required, or its Root, must be had by Approximation only, or by an infinite series. In which Cases, we must have recourse to the general Solution. The Precepts for this particular Solution are these: *viz.*

Strike out the fluxionary Letter, add Unity to the Exponent of the flowing Quantity, in the Expression, and divide by that Exponent thus increased by Unity.

These must be done for every one of the flowing Quantities in the given Equation. Thus the Fluent of the Expression $a^3x\dot{x} \mp bz^3\dot{z} \mp y^4\dot{y}$ is $\frac{1}{2}a^3x^2 \mp \frac{1}{4}bz^4 \mp \frac{1}{5}y^5$. We may be assured it is right, if from the Fluents found, we return to the Fluxions given.

Here it was that Sir *Isaac* had occasion to muster up so much Force of Intellect, to exert such Efforts of amazing Sagacity, such Subtilty of Invention, such Variety of analytical Expedients, as far surpassed all former Experience, for moulding and forming the most crabbed Expressions of mathematical Quantity, into such familiar Shapes and Transformations, as would render it equally manageable, as by the few and easy Rules of his direct Method.

When this particular Solution will not take place, and there are only two flowing Quantities in the Equation with their Fluxions, it is required that the Equations be reduced to such a Form, as that on one Side may be had the Ratio of the Fluxions, (as $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{z}}{\dot{x}}$, &c.) and on the other Side the

Value

Value of that Ratio, expressed by simple algebraic Terms. The Antecedent of the Ratio, or its Fluent, will be the Quantity to be extracted, and the Consequent for the greater Simplicity may be made Unity. Thus the Equation $2\dot{x}+2x\dot{x}-y\dot{x}-\dot{y}=0$ is reduced to this $\frac{\dot{y}}{\dot{x}}=2+2x-y$. So the Equation $\dot{y}a-\dot{y}x-\dot{x}a+\dot{x}x-\dot{x}y=0$, making $\dot{x}=1$, will become $\dot{y}=\frac{a-x+y}{a-x}=1+\frac{y}{a-x}$; when in the Value of the Ratio thus obtained any Term is denominated by a compound Quantity, or is radical; or if that Ratio be the Root of an affected Equation, the Reduction must be performed either by Division, or Extraction of Roots, or by the Resolution of an affected Equation. If in the Expression above we reduce the Term $\frac{y}{a-x}$, denominated by the compound Quantity $a-x$ to an infinite Series of simple Terms $\frac{y}{a}+\frac{xy}{a^2}+\frac{x^2y}{a^3}\frac{x^3y}{a^4}$ *&c.* by dividing the Number y by the Denominator $a-x$, we shall have $\frac{\dot{y}}{\dot{x}}=1+\frac{y}{a}+\frac{xy}{a^2}+\frac{x^2y}{a^3}+\frac{x^3y}{a^4}$, &c. by the help of which the Relation between x and y is to be determined.

For the sake of Perspicuity, and to fix the Imagination, Sir *Isaac* introduces a Distinction of Fluents and Fluxions, into *Relate* and *Correlate*. The Correlate is that flowing Quantity, which he supposes to flow equably, and is given, or may be assumed at any Instant of Time, as the known Measure or Standard to which the Relate Quantity may be always compared. It may therefore very properly denote Time; and its Velocity or Fluxion, being an uniform and constant Quantity, may be made the fluxional Unit; or the known Measure of the Fluxion (or of the rate of flowing) of the Relate Quantity.

Quantity. The Relate Quantity or Quantities is that or those, which are supposed to flow unequably, with any Degrees of Acceleration or Retardation; and their Inequability may be measured or reduced, as it were to Equability, by constantly comparing them with corresponding Correlates or equable Quantities. This therefore is the Quantity to be found by the Problem, or whose Root is to be extracted from the given Equation. And it may be conceived as a Space described by the inequable Velocity of a Body or Point in motion, while the equable Quantity, or the Correlate, represents or measures the Time of Description. This may be illustrated by our common mathematical Tables of Logarithms, Sines, Tangents, Secants, *&c.* In the Table of Logarithms, for Instance, the Numbers are the correlate Quantity, as proceeding equably, or by equal Differences; while their Logarithms, as a relate Quantity, proceed inequably, and by unequal Differences. So the Arches or Angles may be considered as the correlate Quantity, because they proceed by equal Differences, while the Sines, Tangents, Secants, *&c.* are as so many relate Quantities, whose Rate of Increase is exhibited by the Tables.

In respect of this Problem, Equations may be distinguished into three Orders; First, in which two Fluxions of Quantities, and only one of their flowing Quantities are involved. Secondly, In which the two flowing Quantities are involved, together with their Fluxions. Thirdly, In which the Fluxions of more than two Quantities are involved.

Solution of Case First.

Suppose the flowing Quantity, which alone is contained in the Equation to be the Correlate, and the Equation being accordingly disposed, (*i. e.* by making on one Side to be only the Ratio of the

3 Fluxion

Fluxion of the other to the Fluxion of this; and on the other Side to be the Value of this Ratio in simple Terms) *Multiply the Value of the Ratio of the Fluxions, by the corrolate Quantity, then divide each of its Terms by the Number of Dimensions, with which that Quantity is there affected*, and what arises will be equivalent to the other Quantity.

So proposing the Equation $\dot{y}\dot{y}=\dot{x}\dot{y}+\dot{x}\dot{x}x^2$, making $\dot{x}=1$, becomes $\dot{y}\dot{y}=\dot{y}+x^2$, and extracting the square Root, it is $\dot{y}=\frac{1}{2}\pm\sqrt{\frac{1}{4}+xx}=\frac{1}{2}\pm$ the Series $\frac{1}{2}+x^2-x^4+22x^6-5x^8+14x^{10}$, &c. Wherefore it will be found, that $\frac{\dot{y}}{\dot{x}}=1+x^2-x^4+2x^6-5x^8+14x^{10}$, &c.

Therefore $\frac{\dot{y}}{\dot{x}}x=x+x^3-x^5+2x^7-5x^9+14x^{11}$, &c.

and consequently $y=x+\frac{1}{3}x^3-\frac{1}{5}x^5+\frac{2}{7}x^7-\frac{5}{9}x^9+\frac{14}{11}x^{11}$, &c. as may easily be proved by the direct Method. 'Tis here that Mr. *Colson* explains an useful Rule, by which an infinite Expression may be always avoided in the Conclusion; he explains and applies the Difficulties and Anomalies in the Solution of the other two Cases, which is chiefly performed by introducing several new and simple Methods of Analysis, and by applying the Inventor's Artifice of the Ruler and Parallelogram to these fluxional Equations: By which means not only the Forms of the Series are determined, and their initial Approximations; but likewise all the Series may be found, that can be derived from the same fluxional Equation; he then gives a very good general Method for resolving all Equations, whether algebraical or fluxional, founded on the Use and Admission of the higher Orders of Fluxions.

Various are the Methods contrived for finding Fluents, according to the Sagacity of the Artist, and as the Nature and Circumstances of the Problem will admit; and do in fact make up the Bulk of those

Treatises, which contain the Doctrine, Preparation, and Application of Fluxions, to which we refer the Reader for practical Rules therein; a minute Detail whereof is inconsistent, both with our Design and Compass. But as we have not yet touched upon that extensive and noble Step for the finding of Fluents, called the Quadrature of Curves, we will conclude this Subject with an Idea of its Nature and Application.

Since the Line AG=BD is constant, and remains the same, while the Ordinate BC flows with a Velocity continually accelerated or retarded in its perpendicular Direction, but uniformly along AB, (see Fig. p. 271, Dissertation 5.) therefore the Areas of the Curve and Parallelogram ABC and ABDG described in the same time, are likewise flowing Quantities, and their Velocities of Description, or their Fluxions, must necessarily be as their respective describing Lines BD and BC, from the very Definition of Fluxions, as is there proved. Let AG or BD be linear Unity, or a constant known right Line, to which all the other Lines are to be compared or referred; just as in Arithmetic, all other Numbers are tacitly referred to 1, or to numeral Unity, as being the simplest of all Numbers. And let the Area ACB be supposed to be applied to BD, or linear Unity; that is, be divided by it, by which means the Area will be reduced from the Order of Surfaces to that of Lines; and let the resulting Line be called z. That is, make the Area ACB$=z\times$BD; and if AB be called x, then is the Area ABDG$=x\times$BD. Therefore the Fluxions of these Areas will be $\dot{z}\times$BD, and $\dot{x}\times$BD, which are as as $\dot{z}$ and $\dot{x}$. But the Fluxions of these Areas were before proved to be as BC to BD. So that it is $\dot{z}:\dot{x}::$ BC : BD$=1$, or $\dot{z}=\dot{x}\times$BC. Consequently, in any Curve, the Fluxion of the Area will be, as the Ordinate of the Curve multiplied in the Fluxion of the Abscis.

Thus

Thus then to apply it, if in the conic Parabola a be the Latus Rectum, y be the Ordinate, and x the Abscifs, then by the Property of the Figure $y^2 = ax$, (and the fame will hold true in general, where $y^m = a^{m-1}x$; or $x^m = ay^{m-1}$) therefore $y = \overline{ax}|^{\frac{1}{2}}$, therefore $y\dot{x}$, by the Doctrine of Quadratures just now laid down, is $= \overline{ax}|^{\frac{1}{2}} = \dot{z}$; therefore the Fluent of this must be the Curve, which by the Rules before laid down, must be ${}^2ax^{\frac{3}{2}} = \frac{2}{3}x \times ax^{\frac{1}{2}}$; that is, $\frac{2}{3}$ of the Abfcifs multiplied into the Ordinate, gives the Area, which we know to be true by other Methods.

By this Method, when we enquire into the Area of any Curve propofed, that Area may be exhibited either *arithmetically*, as above, or *geometrically*, by finding and defcribing other more fimple Curves with which it may be compared. The great Inventor has conftructed a Table or Catalogue of Curves, that are capable of being compared geometrically with the Ellipfe and Hyperbola, fo that their Areas may be exhibited by the Defcription of thefe Figures; and confequently given, when thefe Figures are given.

We might produce a Multiplicity of Examples to verify the above Doctrine of the inverfe Method of Fluxions, but the above is fufficient to prove the Rationale of it.

This fecond general Problem branches itfelf out to a great many noble and ufeful Purpofes in Mechanics, Aftronomy and Phyfics; fuch as the Rectification of curve Lines, the plaining of curve Surfaces, the Cubature of Solids, the finding Centers of Gravity and Percuffion of all Lines, Surfaces and Bodies; and, in fhort, to the whole Science of Motions.

And now we hope our Defign is compleated, efpecially to the Satisfaction of the Candid; which was to fhew, what the modern Improvements in Geome-

try were; and that they were founded upon the ſame Principles, and deduced with as much Accuracy, but extended much farther than thoſe of the Antients; that they are intirely ſcientific, and thoroughly freed from any Trick or Quirk, as has been inſinuated by the ingenious Author of the Analyſt.

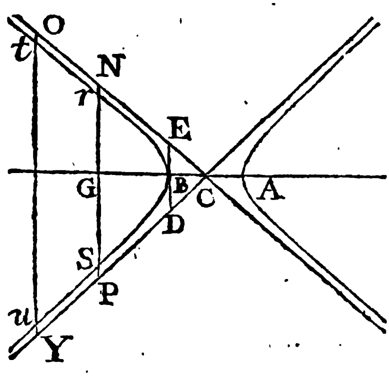
CONIC

CONIC SECTIONS.

Of the Hyperbolic Aſſymptotes.

PROPOSITION XLII.

IF any Ordinate (NGP) to the Axe be continued both ways to the Aſſymptotes, then the Square of the Semiconjugate Axe (BE) will be equal to the Rectangle of the greateſt, and leaſt Diſtance of either Extremity of that Line from the Curve; *that is*, $BE^2 = NS \times SP = Pr \times rN$.



DE-

DEMONSTRATION.

Let NG=PG=b, and the other Symbols as usual; then CG=$\frac{1}{2}t+x$, and (*by Sim. Trian.*) CB^2 : BE^2 : : CG^2 : GN^2; that is, $\frac{1}{4}t^2 : \frac{1}{4}tp$ $(=\frac{1}{4}c^2)$: : $\frac{1}{4}t^2+tx+x^2 : b^2$; therefore $b^2=\frac{p}{t}\times\overline{\frac{1}{4}t^2+tx+x^2}$. But (*by Prop.* 2.) $t:p::tx+x^2:y^2$, therefore $y^2=\frac{p}{t}\times\overline{tx+x^2}$, and $b^2-y^2=\frac{1}{4}c^2$ $(=\frac{p}{t}\times\frac{1}{4}t^2=\frac{1}{4}pt.)$ Also $b+y:\frac{1}{2}c::\frac{1}{2}c:b-y$, or NS : EB : : EB : SP. Q. E. D.

PROPOSITION XLIII.

The Assymptotes continually approach to the Curve.

DEMONSTRATION.

By the 42d, EB^2=NS×SP=Oa×aY, therefore NS : Oa : : aY : SP. But NS is less than Oa, therefore aY is less than SP, and consequently the Point Y is nearer to the Curve than the Point P. Q. E. D.

PROPOSITION XLIV.

If the Assymptotes and Curve be infinitely produced, they will never concur.

DEMONSTRATION.

From the two first Analogies of Proposition 42, it follows, that $\overline{\frac{1}{2}t+x})^2 : b^2 : : \overline{t+x}\times x : y^2$; that is, $CG^2 : GN^2 : : AG\times BG : Gr^2$. But (*by* 6 *E.* 2) CG^2 is greater than AG×BG, therefore GN^2 is greater than Gr^2, and GN greater than Gr; consequently wherever the Point N is taken, it will never touch the Curve.

PROPO-

PROPOSITION XLV.

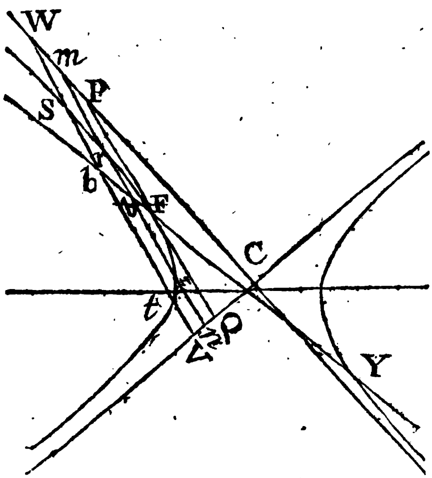
If an Ordinate to the Axe be produced both ways to the Assymptotes, then the Parts intercepted on each Side between the Curve and Assymptotes are equal; that is, $SP = rN$.

DEMONSTRATION.

From similar Triangles $BD : BE :: GP : GN$; but $BD = BE$, therefore $GP = GN$, and the Ordinate GS, Gr being equal, rN will be $= SP$. Q. E. D.

DEFINITION.

If the Tangent to the Vertex of any Diameter be continued both ways from the Point of Contact, with



this Condition, that as the Diameter passing through the Point of Contact, is to its Parameter, so is the Square of the Semidiameter to a fourth Proportional; then if the square Root of that fourth Proportional be set both ways from the Vertex on the Tangent, (as FP, FQ) the Extremities will determine

mine the conjugate Diameter; and if through these Extremities right Lines (as CP, CQ) be drawn from the Center, they shall be Assymptotes.

Proposition XLVI.

If any Ordinate (as *mbn*) to a Diameter be produced both ways to the Assymptotes, then the Square of the semi-conjugate Diameter will be equal to the Rectangle of the greatest and least Distance of either Extremity of that Line from the Curve; that is, $FP^2 = mz \times zn = nr \times rm$.

Demonstration.

Put $bm = r$, $br = y$, $FP = FQ = \frac{1}{2}c$; then (*by Def.*) $D : P :: CF^2 : FP^2$; or $D : P :: \frac{1}{4}D^2 : \frac{1}{4}c^2 = \frac{\frac{1}{4}PD^2}{D} = \frac{1}{4}PD$. But (*by Sim. Trian.*) $CF^2 : FP^2 :: Cb^2 : bm^2$; *that is*, $\frac{1}{4}D^2 : \frac{1}{4}DP :: \overline{\frac{1}{2}D + x}^2 : r^2$, therefore $r^2 = \frac{P}{D} \times \overline{\frac{1}{4}D^2 + Dx + x^2}$, and (*by Prop.* 30.) $D:P :: \overline{D+x} \times x : y^2$, therefore $y^2 = \frac{P}{D} \times \overline{Dx + x^2}$, and $r^2 - y^2 = \frac{1}{4}c^2 \left(\frac{P}{D} \times \frac{1}{4}PD^2.\right)$ Also $r + y : \frac{1}{2}c :: \frac{1}{2}c : r - y$; or $FP^2 = mz \times zn = rn \times rm$. Q. E. D.

Proposition XLVII.

The Assymptotes, drawn thro' the Extremities of any conjugate Diameter, and produced, do continually approach to the Curve.

Demonstration.

By Proposition 46, $mz \times mr =$ ($FP^2 =$) $wt \times ws$, therefore $mz : wt :: ws : mr$. But mz is less than wt; therefore ws is less than mr, and consequently the

the Point w is nearer the Curve than the Point m. Q. E. D.

PROPOSITION XLVIII.

The Assymptotes, produced thro' the Extremities of the conjugate Diameter, will never meet the Curve.

DEMONSTRATION.

By the 46th, $\frac{1}{4}PD = \frac{1}{4}c^2 = FP^2$; and (*by sim. Trian.*) $\overline{bw}^2 : \overline{Cb}^2 :: (FP^2 : CF^2 :: \frac{1}{4}PD : \frac{1}{4}D^2 ::) P : D$, also (*by Proposition* 30.) $\overline{bs}^2 : Yb \times Fb :: P : D$; therefore (*by Equality*) $\overline{bs}^2 : Yb \times Fb :: \overline{bw}^2 : \overline{Cb}^2$. But (by 6. E. 2) $\overline{Cb}^2$ is greater than $Yb \times Fb$, therefore $\overline{bw}^2$ is greater than $\overline{bs}^2$, and bw greater than bs; consequently whenever the Point w be taken, in CP produced, it will be without the Curve. Q. E. D.

PROPOSITION XLIX.

If Ordinates to any Diameter be produced both Ways to the Assymptotes; then the external Parts between the Assymptotes and the Curve are equal; *that is*, $rm = zn$.

DEMONSTRATION.

By sim. Trian. $PF : FQ :: bm : bn$. But $PF = FQ$, therefore $bm = bn$, from which, if you take away the equal Ordinate, there will remain $rm = zn$. Q. E. D.

PROPOSITION L.

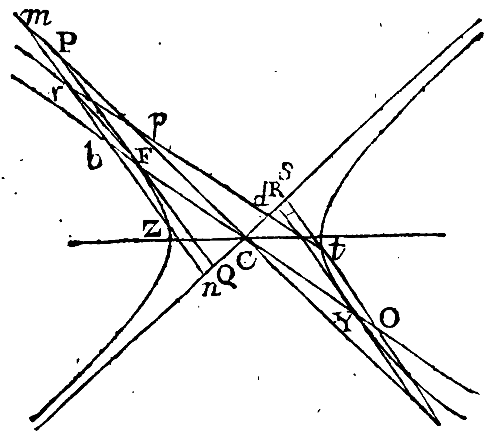
If the right Line rt be drawn parallel to the Diameter FY, then the Square of the Semi-diameter CF shall be equal to the Rectangle contained under the greatest and least Distance of either Extremity

 of

of that Line from its adjacent Assymptote; *that is*, $CF^2 = rd \times rp = pt \times td$.

DEMONSTRATION.

Put $FQ = PF = c$, $FC = t$, $mr = b$, $rp = d$, $rn = p$, and $rd = q$; then, because the Triangles mrp, PFC,



and likewise nrd, QFC, are similar (by 4. E. 6.) $\frac{b}{d} = \frac{c}{t}$, and $\frac{p}{q} = \frac{c}{t}$; therefore $\frac{pb}{qd} = \frac{c^2}{t^2}$, or $pb : c^2 :: qd : t^2$. But (*by the* 46) $bp = c^2$, therefore $qd = t^2$, or $CF^2 = rd \times rp$. Q. E. D.

PROPOSITION LI.

If a right Line be drawn parallel to any Diameter, and cut the opposite Hyperbolas; then the Parts of that Line intercepted between the Curves and Assymptotes are equal; *that is*, $rp = td$.

DEMONSTRATION.

Make the Abscissa, $Yo = Fb$; draw the Ordinate ot, and the Conjugate YR; then (*by Sim. Trian.*)

mr

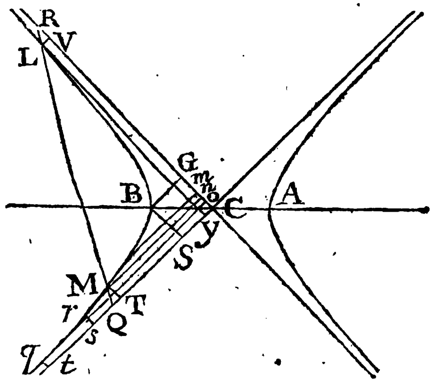
$mr : rp :: (PF = FQ : FC :: YR : YC ::)\ St : td.$
But $mr =$ $(zn =)$ St, therefore $rp = td$. Q. E. D.

Proposition LII.

If, through any two Points (L, M) in the Curve, right Lines (LV, MT) be drawn parallel to the Assymptote; then, the Rectangles under each of these Lines, and the adjacent Distance (on the Assymptote) from the Center, shall be equal; *that is*, $LV \times VC = MT \times TC$.

Demonstration.

Through the Points L, M, draw the right Line LM, and let $RL = y$, $LV = d$, $RV = p$, $MQ = z$, $QT = x$,



$MT = c$, $VC = b$, and $TC = a$; then, by reason of the Parallels, the Triangles RVL, RCQ, MTQ, are similar. But (*by* 49.) $y = z$, therefore $c = p$, and $x = d$, whence $c\ (p) : d :: (p + b)\ c + b : (a + x)\ a + d$, therefore $ca = db$, or $LV \times VC = TM \times TC$. Q. E. D.

Corollary I.

Hence, if the Lines MT, *rs*, *qt*, &c. be drawn parallel to the Assymptote CR, and the Parallelograms *Tm*, *Sn*, *to*, &c. be inscribed, they will be

 equal

equal to each other. Because, by the same Reasoning, as in this Proposition, we may prove each of them equal to the Parallelogram LC.

COROLLARY II.

Each of the inscribed Parallelograms T*m*, S*n*, &c. is equal to the Square of a right Line (as BS) drawn from the Vertex B, parallel to the Aymptote CR. For *(by this Proposition)* each of them is equal BS (=GC) ×SC; but *(by the Genesis)* the Angle BCG = the Angle BCS, and *(by Parallels)* the Angle BCG = the Angle SBC, therefore BCS=SBC, and *(by 6. E. 1.)* BS = SC; consequently each of the Parallelograms T*m*, S*n*, &c. is equal to BS× BS, or BS^2.

SCHOLIUM.

Right Lines (as *tq*, *sr*, &c,) drawn from one Assymptote, parallel to the other, and terminated by the Curve, are called Ordinates; the Distance of those Lines from the Center (as *t*C, *s*C) Abscissas; and a right Line (as BS) drawn from the Vertex parallel to the Assymptote, the Parameter of the exterior Hyperbola; also if p be put for such Parameter, x for the Abscissa, and y for the Ordinate; then *(by the last Corol.)* $pp=yx$.

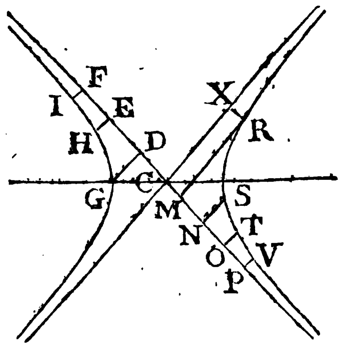
PROPOSITION LIII.

If on either of the Assymptotes (as CF) from the Center right Lines (as CD, CE, CF) be set off in continual Proportion; and if, from the Extremities of these Lines, there be drawn Lines (as DG, EH, FI) parallel to the other Assymptote, and continued to the Curve, they shall likewise be in continual Proportion; *that is*, if CD, CE, CF be in continual Proportion, then DG, EH, FI will be in continual Proportion.

DEMON-

DEMONSTRATION.

By the 52, $GD \times DC = HE \times CE$, and $CF \times FI = CE \times EH$, also (*by Supposition*) $CD \times CF = CE \times CE$,



therefore $DG : HE :: (CE : DC :: CF : CE ::)$ $EH : FI$. Q. E. D.

PROPOSITION LIV.

If, on either Assymptote, there be set off equal Parts from the Center; *that is*, if right Lines be set off from the Center in continual arithmetical Proportion, (as CM, CN, CO, CP, *&c.*) and from the Extremities of these, there be drawn right Lines, (as MR, NS, OT, PV) parallel to the other Assymptote, and continued to the Curve, these shall be in continued harmonic Proportion.

DEMONSTRATION.

$MR\ (CX) \times CM = NS \times CN = OT \times CO = VP \times CP$, therefore

$$\left.\begin{matrix} CM : CN :: NS : MR \\ CM : CO :: TO : MR \\ CM : CP :: VP : MR \end{matrix}\right\} ;\ \text{but}\ CM = \left\{\begin{matrix} \frac{1}{2}\,CN \\ \frac{1}{3}\,CO \\ \frac{1}{4}\,CP \end{matrix}\right\}$$

$$\text{therefore} \left\{\begin{matrix} NS \\ TO \\ VP \end{matrix}\right\} = \left\{\begin{matrix} \frac{1}{2} \\ \frac{1}{3} \\ \frac{1}{4} \end{matrix}\right\} MR\ ;\ \text{and if}\ MR = 1,\ \text{then}$$

NS

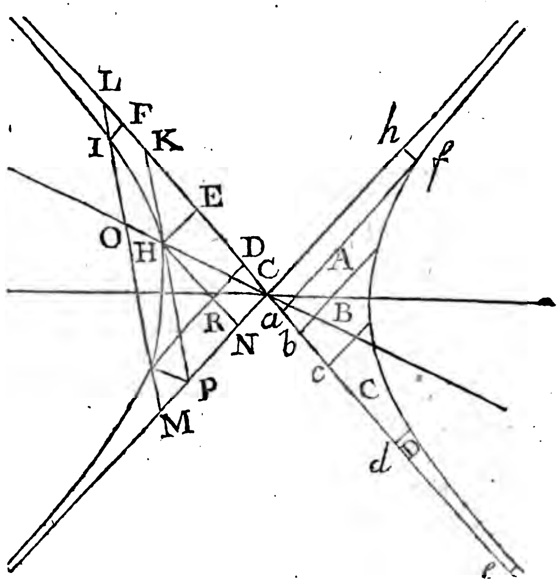
NS$=\frac{1}{2}$, OT$=\frac{1}{3}$, VP$=\frac{1}{4}$; which (being the Reciprocals of continued arithmetical Proportion) are in continued harmonic Proportion. Q. E. D.

Proposition LV.

If from the Center on either Aſſymptote, there be ſet three continual Proportionals (as CD, CE, CF) and from their Extremities right Lines (as DG, EH, FI) be drawn parallel to the other Aſſymptote, and continued to the Curve; alſo, if on the Curve through the Ends (as I, G) of the Extremes a right Line, as LM, be drawn, then I ſay a right Line, drawn from the Center through (H) the End of the Mean, ſhall biſect that Line; *that is*, CO biſects IG in O.

Demonstration.

Draw HK parallel to LM; then (*by ſim. Trian.*) EH : KH :: FI : LI; and EH : KH :: DG : GL;



therefore

therefore $EH^2 : KH^2 :: FI \times DG : LI \times GL$. But (*by the* 53) $EH^2 = FI \times DG$, therefore $HK^2 = LI \times GL$, and (*by the* 46) KH is a Tangent to the Point H; consequently IO (=OG, being parallel to it) is an Ordinate to the Diameter CO. Q. E. D.

Proposition LVI.

If CD, CE, CF on the Assymptote (and consequently, *by the* 53, DG, EH, FI) be in continual Proportion; then the Spaces (HEDG, EHFI) between the Curve and Assymptotes on each Side of the Mean (EH) to the Extremes (FI and DG) shall be equal.

Demonstration.

1. Through the Points I and G, draw the right Line LM, and, through the Center and H, draw the right Line CO; then (*by Prop.* 55 *and* 49) LO =OM, therefore (by 1. E. 6.) the Trian. MOC= Trian. OCL. But the Space OGH = the Space OHI, because each is composed of an indefinite Number of equal Ordinates, consequently the Space CHGM = Space CHIL; and taking away from each the Triangles MGP+NHC=Triangles FLI +HCE, there remains the Space NHGP = Space EHFI.

2. But the Parallelograms CG and EN are equal by 52, therefore the Parallelograms NG, RC, and consequently the Spaces HEDG, EHFI (*equal by the first Part*, Space NHGP = Space HRG+ Parall. RE = Space HRG + Parall. NG) are equal. Q. E. D.

Proposition LVII.

If on either Assymptote be set off continual Proportionals, and from their Extremities right Lines be drawn parallel to the other Assymptote, then the Spaces between these Lines shall be as the Logarithms of

of the Ratio's of the Lines which bound them. *That is*, if Ca, Cb, Ce, Cd, *&c.* be in continual Proportion, then the Space *ackf* is as the Logarithm of the Ratio of *ck* to *af*; and the Space *adgf*, as the Logarithm of the Ratio of *dg* to *af*, &c.

DEMONSTRATION.

Let the Spaces between the Parallels be *A*, *B*, *C*, *D*, &c. (*as in the Figure*) then (*by Supposition*) $\frac{Ca}{Cb}=\frac{Cb}{Cc}$, therefore (*by Prop.* 56.) $A=B$, and $\frac{Ce}{Cd}=\frac{Cd}{Ce}$, therefore $B=C$, &c. that is, if $\frac{Ca}{Cb}=\frac{Cb}{Cc}=\frac{Cc}{Cd}=\frac{Cd}{Ce}$ &c. then $A=B=C=D$, &c. whence the Spaces are a Series of continued arithmetical Proportionals, fitted to a Series of continued geometrical Proportionals; and consequently the Addition of one answers to the Multiplication of the other, which is the Property of Logarithms. *As for Example.*

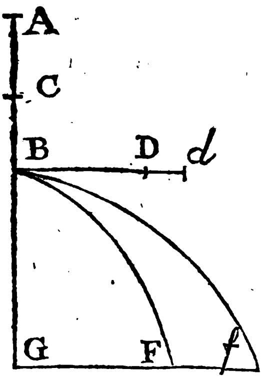
Multiply the geometrical Series $\frac{Ca}{Cb}=\frac{Cb}{Cc}$, the Product will be $\left(\frac{Ca}{Cc}= \textit{by Prop. } 52\right)\frac{ck}{af}$; and add the corresponding arithmetical Series, and the Sum is $(A+B=)$ the Space *ackf*; consequently the Space *ackf* is as the Logarithm of the Ratio of *ck* to *af*, Q. E. D.

PROPOSITION LVIII.

The Areas of the two Hyperbola's, having the same transverse Axis, are as their Conjugates.

DEMON-

DEMONSTRATION.



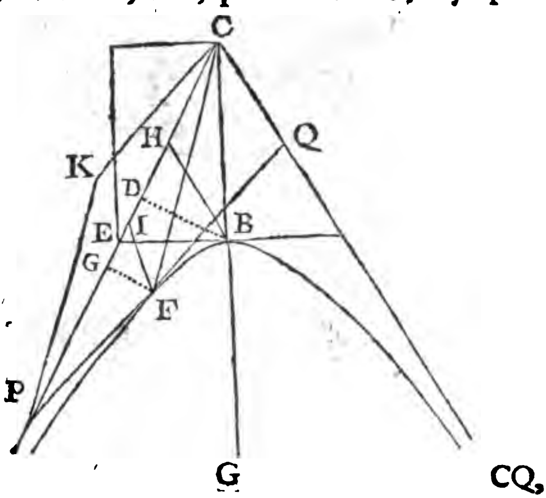
Let FB, *f*B be two Hyperbola's described to the same transverse Axis AB; then (*by Prop.* 1.) $GF^2 : BGA :: BD^2 : BC^2$, and $Gf^2 : BGA :: Bd^2 : BC^2$; therefore (*by Equality*) $GF^2 : Gf^2 :: BD^2 : Bd^2$, and (*by* 22 *E.* 6.) $GF : Gf :: BD : Bd$. But the Sum of all the GF, G*f* do respectively constitute the Areas of the hyperbolic Spaces BFG, B*f*G; therefore (*by* 12 *E.* 5.) those Areas are as the conjugate Axes. Q. E. D.

PROPOSITION LIX.

Parallelograms circumscribing any Diameters of an Hyperbola are equal.

DEMONSTRATION.

From the Vertex of the Diameter, and of the Curve, draw FI, BH, parallel to the Assymptote



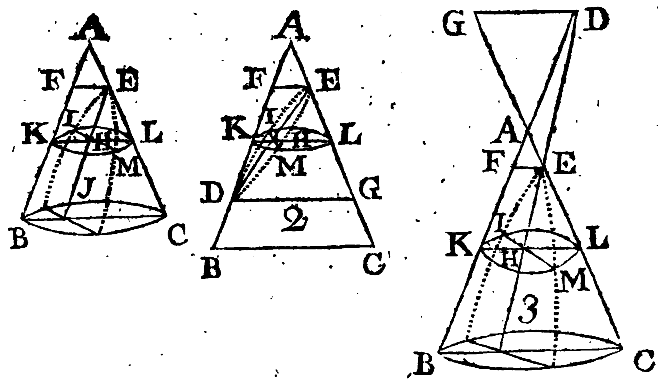
CQ, and to the other Aſſymptote let fall the Perpendiculars FG, BD. Put $IC=x$, $FI=y$, $BD=c$, and $CH=a$, then (*by ſim. Trian.*) FP : FQ :: IP : IC; but PF=FQ; therefore PI=IC, and $CP=2x$, alſo FG (*by the ſim. Trian.* HBD, GIF, is) $=\frac{cy}{a}$; whence the Area of the Parallelogram PFCK = PC $\times FG=\frac{2cyx}{a}=$ (becauſe by *Scholium* to *Prop.* 52. $yx=a^2$) $2ac=CE\times BD=EBC$. Q. E. D.

APPENDIX.

APPENDIX.

The Properties of the *Parabola*, *Ellipſe* and *Hyperbola*, made by cutting a Cone by a Plane.

LET ABC be a Cone, ſtanding on a circular Baſe BC, and IEM a Section thereof, made by a Plane inclined to the Baſe; let KILM be any other Section parallel to the Baſe, and meeting the former Section in HI; alſo, let ABC be a third



Section, perpendicularly biſecting the former in EH and KL reſpectively, and the Cone in the Triangle ABC; then producing EH (*Fig.* 3.) till it meet AK in D, draw EF and DG, parallel to KL, meeting AB

and AC in F and G, and call EF, a; and DG, b; ED, c; the Abscissa EHx; and the Ordinate HI, y; then, by reason of the similar Triangles EHL, EDH, we have ED : DG :: EH : HL $=\frac{bx}{c}$; also, because of the similar Triangles DEF, DHK, DE : EF :: DH ($c-x$ in Fig. 2, and $c+x$ in Fig. 3.) HK $=\frac{ac\mp ax}{c}$; lastly, since the Section KIL is parallel to the Base, and consequently circular, HK$\times$KL will be $=$ HI^2; *that is*, $\frac{abcx\mp abx^2}{c^2}=y^2$, and if p be a fourth Proportional to c, a and b, then $\frac{ab}{c}=p$, and (by Substitution) $\frac{pcx\mp px^2}{c}=y^2$; also, if X and Y be put for any other Abscissa and Ordinate; then, by the same Reasoning, $\frac{pcX\mp pX^2}{c}=Y^2$. Hence,

1. When a Cone is cut by a Plane, which intersects both its Sides (*as in Fig.* 2.) then the Property of the Curve, made by the Plane of that Section, will be such, that $\overline{c-x}\times x : y^2 :: (c:p::) \overline{c-X}\times X : Y^2$, which is the same Property with that in Corollary to Proposition 2. of the Ellipse foregoing.

2. When a Plane cuts the Base and Side of a Cone continued from the Vertex, (*as in Fig.* 3.) the Property of the Curve, made by the Plane of that Section, will be such that $\overline{c+x}\times x : y^2 :: (c:p::) \overline{c+X}\times X : Y^2$, which is the same Property as that in Corollary to Proposition 2. of the Hyperbola preceding.

3. If

3. If a Cone be cut by a Plane parallel to one of its Sides (*as in Fig.* 1.) and if AF$=a$, HK$=b$, the Abſciſſa EH$=x$, and the Ordinate IH$=y$, then (*by ſim. Trian.* AFE, EHL) AT: (FE) KH : : EH : HL$=\frac{bx}{a}$, but KH$\times$HL$=$HI^2; that is, $\frac{b^2x}{a}=y^2$, and making p a third Proportional to a and b, we have $\frac{b^2}{a}=p$, and (*by Subſtitution*) $px=y^2$; alſo putting X and Y for any other Abſciſſa and Ordinate, by the ſame manner of Reaſoning $pX=y^2$, whence $Y^2 : y^2 : :$ (pX $: px : :$) $X : x$; which is the ſame with Cor 1. Prop. 1. of the Parabola.

ANSWERS

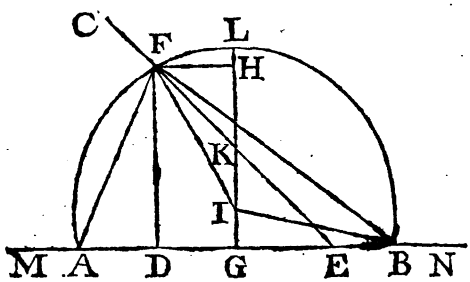
ANSWERS TO THE PROBLEMS

Propofed in the Fifth NUMBER.

PROBLEM LXXVII. *Anfwered by* John Turner.

CONSTRUCTION.

DRAW the indefinite right Line MN, in which take AB eqnal to (80) the given Bafe, and AE equal to (68) the given Sum; upon AB let a Segment of a Circle be de-



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ſcribed to contain the given Angle ($=84°$) and make the Angle AEC equal to half a right Angle; from F, the Point where EC cuts the Arch of the Circle, to AB let the Perpendicular FD be drawn; join A, F and B, F then AFB will be the Triangle required.

DEMONSTRATION.

Since AEF equal to half a Right-Angle, and FD perpendicular to AB (*by Conſtr.*) it is evident, that $DF=DE$, and conſequently $AD+DF=AD+DE=AE$, the given Sum, by Conſtruction. W. W. D.

CALCULATION.

Thro' I, the Center of the Circle, perpendicular to AB draw LG, cutting EC in K; join B, I and F, I and draw FH parallel to AB.

In the Triangle BIG are given all the Angles and the Side BG ($=\frac{1}{2}AB$) whence GI and BI are found equal to 4. 2 and 40. 2 reſpectively; then in the Triangle FIK are given FI, $IK=GK$ (GE) $-$ GI and the Angle K, whence IFK will be found $=24°, 8'$, which added to KFH ($=45°$) gives the Coſine of the Difference of the Angles at the Baſe; from whence the Angles themſelves, and likewiſe the Sides, may be found.

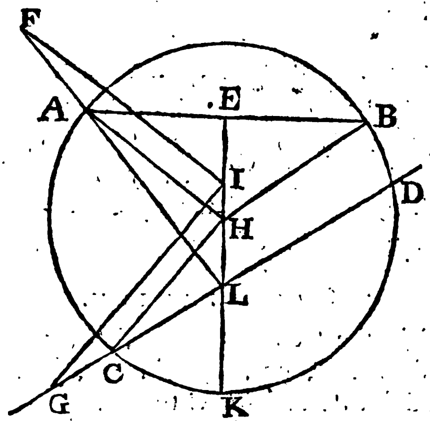
PROBLEM LXXVIII. *Anſwered by Mr.* Sam. Clark.

Let A and B be two given Points, and GD the Line given in Poſition.

CON-

Construction.

Join A and B, bisect AB in E, draw EK perpendicular to AB; from the Point of Intersection



L draw LA, which produce towards F; from any Point in EK as I, draw IG making an Angle with DG, equal to what the given one is either above or under 90 Degrees; make IF=IG, from A draw AH parallel to FI, and H will be the Center of the Circle required.

Demonstration.

Draw CH parallel to GI, and join the Points H, B the Triangles FIL, AHL, are similar, as also are the Triangles IGL, HCL, whence we have FI : IL : : AH : HL, and GL : CL : : IL : HL. Consequently FI : AH : : IG : CH, but FI=IG (by Construction) therefore AH=HC=AB.

Method of Calculation.

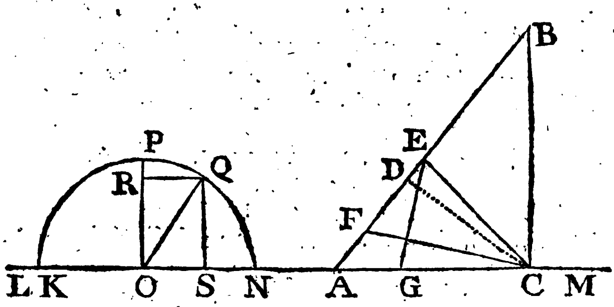
From the given Position of GD the Line EL and Angle ELG may be found, whence the Hypothenuse

thenuse AL becomes known; then in the Triangle IGL we have all the Angles, and a Side (IL being taken at pleasure) given to find IG = IF, and in the Triangle IFL we have sufficient to find FL, then it will be FL : FI : : AL : AH, the Radius required. Q. E. I.

PROBLEM LXXIX. *Answered by* John Turner.

CONSTRUCTION.

Draw the indefinite Right Line LM, in which take AK equal to (100) the given Hypothenuse,



and AC equal to (80) the given Base; take NK to NA, as (1680) the Sum of the Areas of the two Extremes to (720) the given Area of the Mean, (*see the Errata*) and upon KN let the Semicircle KPN be described; bisect KN with the Perpendicular OP, in which take OR = NA; draw RQ, parallel to LM, meeting the Semicircle in Q, and draw QS perpendicular to LM. Upon AC let the right-angled Triangle ABC be constructed, whereof the Side BC is equal to (60) the given Perpendicular, and the Hypotheneuse AB equal to AK; make AF = NS, BE = KS, and join C, F, and C, E; then take AG a fourth Proportional to AE, AF, and AC, and join E, G, and CE, EG will be the Lines required.

DEMONSTRATION.

Draw the Perpendicular CD.

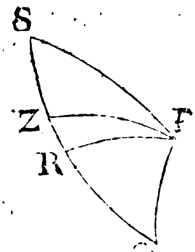
Becauſe AK=AB, KS=BE, and SN=AF, it is evident that NA=FE; but, by the Property of the Circle, (NA (=SQ) is a geometrical Mean between KS and SN, therefore the Triangles FCA, FCE, and ECB having the ſame Altitude CD are to one another as their reſpective Baſes; conſequently the Triangle FEC a geometrical Mean between the other two. Moreover, (by Conſtruction) AG is a fourth Proportional to AE, AF, and AC, therefore AE×AG=AF×AC, or the Triangle ACF=AEG, conſequently CEG=CFE. W. W. D.

CALCULATION.

Join O, Q.

By ſim. Triangles AB : CB :: AC : CD=48, whence (the Area of the Mean being given) the Baſe FE will be found =30, conſequently AF+EB = KN = 70; then in the Triangle QOS are given OQ and QS, whence OS will be found = 18.0277563, KS (=BE) =53.0277563, and SN (=AF) =16.9722437; therefore AG=28.9.

PROBLEM LXXX. *Anſwered by* John Turner.

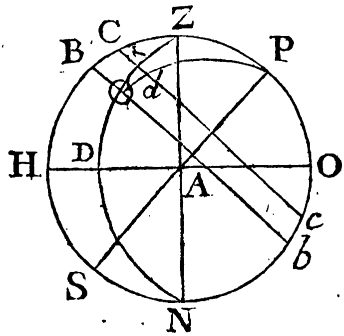


Let P be the Pole, Z the Zenith of the given Place, and QZS an Arch of a great Circle paſſing thro' the Zenith, and interſecting the Arches PS, PQ in S and Q; biſect SQ with the Perpendicular PR, then it will be as Radius to the Coſine of RPS ($\frac{1}{2}$SPQ) ſo is the Tangent of SP to the Tangent of RP=32°, 47′, and as Sine of ZP to the Sine of RP, ſo is Radius to the Sine of

RZP the Angle required, which is equal to 60°, 31′.

PROBLEM LXXXI. *Answered by* John Turner.

Let HZON be an orthographic Projection of the Sphere, in which PZ represents the Complement of



Latitude, PC and PB the Complements of the two Declinations, B*b*, and C*c*, the Parallels of the two Declinations, and ZDN an Azimuth Circle; then, the Arch ($d\odot$) of the Meridian included between the two Parallels being always the same, it will appear that the Arch $r\odot$ of the Azimuth, will be a *Maximum*, when the Angle P⊙Z is a *Maximum*. But (*per Spherics*) *S*. P⊙ : *S*. PZ : : *S*. PZ⊙ : *S*. P⊙Z, whence it is evident that the Angle ⊙ will be a *Maximum*, when the Sine of PZ⊙ is a *Maximum*; that is, when the Angle PZ⊙ is a right Angle.

PROBLEM LXXXII. *Answered by* John Turner.

Let $AH=GK=50=a$, $AI=20=b$, $GH=30=c$, $BC=10=d$, and $BF=x$; then $a:c::x:\frac{cx}{a}=FG$, and the Content of the Solid described by the Triangle BFG (p being put $=3.1416$) will be $pcx^2+\frac{2pc^2x^3}{3a^2}$; also the Content of the Solid de-

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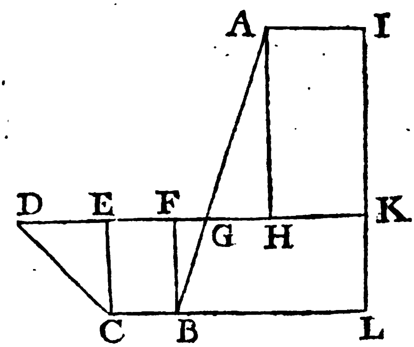
scribed by the Parallelogram BCEF will be $2padx+\frac{2pcdx^2}{a}+pd^2x$, and that of the Solid described by



the Triangle DCE$=pax^2+\frac{pcx^3}{a}+pdx^2+\frac{px^3}{3}$; consequently the Sum of these Solids equal to 204204, the Content of the given Frustum, whence x may be found.

PROBLEM LXXXIII. *Answered by* John Turner.

Let W represent the greater, and w the lesser Weight, then $W-w$ is the Force by which both Weights are accelerated; therefore $W+w : W-w :: W : \frac{W\times\overline{W-w}}{W+w}$ the Force by which the greater is accelerated. But the Distances descended in equal Times are as the Forces, therefore $W : \frac{W\times\overline{W-w}}{W+w} :: 1 : \frac{W-w}{W+w} ::$ so is the Distance a Body will freely descend from Rest in the given Time to 5.36, the Distance required.

PRO-

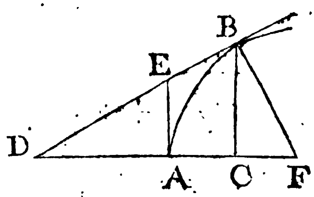
PROBLEM LXXXIV. *Answered by Mr.* Sam. Clark.

In all Curves $\frac{y\dot{x}}{\dot{y}}$ is an Expression for the Subtangent, where x is any Abscis, and y its corresponding semi-ordinate, (and 47, *E*, 1st) the Tangent will be expressed by $\sqrt{\frac{yy\dot{x}\dot{x}}{\dot{y}\dot{y}}+yy}$ whence by the Question we have this Proportion, *viz.* $\frac{y\dot{x}}{\dot{y}}-x:x::\sqrt{\frac{yy\dot{x}\dot{x}}{\dot{y}\dot{y}}+yy}:y$ multiply Extreams and Means, $\frac{yy\dot{x}-xy\dot{y}}{\dot{y}}=\sqrt{\frac{y^2x^2\dot{x}^2+y^2x^2\dot{y}^2}{\dot{y}^2}}$ and by squaring each Side of the Equation we have $y^4\dot{x}^2-2xy^3\dot{x}\dot{y}+x^2y^2\dot{y}^2=y^2x^2\dot{x}^2+y^2\dot{y}^2x^2$, hence $y^2\dot{y}\dot{x}-2x\dot{y}^2y\dot{y}xx\dot{x}$ and $y^2\dot{x}-2xy\dot{y}=xx\dot{x}$, the Fluent of which is $-\frac{yy}{x}$ but as a negative Quantity cannot be equal to an affirmative, it is evident there is some constant Quantity wanting to compleat the Fluent, let it be (a) then $ax-yy=xx$, whence the Curve sought is a Circle, whose Diameter is any right Line at pleasure, and is here expressed by the invariable Quantity (a). Q. E. J.

The

The same, Answered by J. Turner.

By the Problem DA : DB :: AC : BC, whence it is evident, that if A, B be joined the Angles DBA



and ABC will be equal: Draw BF perpendicular to DB and EA to DF, then by reason of the parallel Lines AE, CB it will appear that the Angle EAB is = ABC = ABE; therefore, since FAE and FBE are each right Angles, it is plain that FAB = FBA, FA = FB, and consequently the Curve required a Circle.

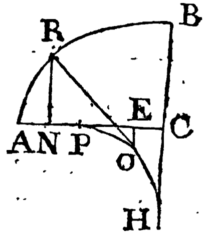
PROBLEM LXXXV. *Answered by* J. Turner.

Let the Altitude of the Curve be represented by x, and the Ordinate by y, then py^2x will be the Content of the Cylinder, and $\frac{3py^2x}{10}$ that of the Solid described by the Rotation of the required Curve about its Axis; therefore its Fluxion $\frac{6pyx\dot{y}+3py^2\dot{x}}{10}$ $=py^2\dot{x}$, the Fluxion of the Cylinder, whence $6\frac{\dot{y}}{y}=$ $=7\frac{\dot{x}}{x}$ consequently the Fluent 6 Log. $y=7$ Log. x, and $x=y^{\frac{6}{7}}$, which substituted in the Expressions of the Solids gives $py^{\frac{20}{7}}$ and $\frac{3py^{\frac{20}{7}}}{10}$.

PRO-

PROBLEM LXXXVI. *Answered by* J. Turner.

Let the Transverse (2AC) be denoted by a, the Conjugate (2BC) by c, the half Parameter (AP) by



$\frac{c^2}{2a}$, the Absciss AN by x, and the Ordinate NR by y; also put PE, the Absciss of the required Curve $= u$, and its Ordinate OE $= w$, then will $w = \frac{4\times\overline{a^2-c^2}\times\overline{ax-x^2}|^{\frac{3}{2}}}{ca^3}$, and $u = \frac{a^2-c^2}{2a} + \frac{\overline{c^2-a^2}\times\overline{a-2x}|^3}{2a^4}$. Put PC $= \frac{a^2-c^2}{2a} = b$, then $u = b - \frac{b\times\overline{a-2x}|^3}{a^3}$ and $w = \frac{8b\times\overline{ax-x^2}|^{\frac{3}{2}}}{ca^2}$; whence $x = \frac{1}{2}a - \frac{1}{2}a \times \overline{\frac{b-u}{6}}|^{\frac{1}{3}}$, $ax - x^2 = \frac{1}{4}a^2 - \frac{1}{4}a^2 \times \overline{\frac{b-u}{b}}|^{\frac{2}{3}} = w|^{\frac{2}{3}} \times \overline{\frac{c}{b}}|^{\frac{2}{3}} \times \frac{\overline{a}|^{\frac{2}{3}}}{4}$; therefore $1 - \overline{\frac{b-u}{b}}|^{\frac{2}{3}} = w|^{\frac{2}{3}} \times \overline{\frac{c}{ab}}|^{\frac{2}{3}}$, and $w = \frac{ab}{c}$ into $\overline{1 - \overline{\frac{b-u}{b}}|^{\frac{2}{3}}}|^{\frac{3}{2}}$.

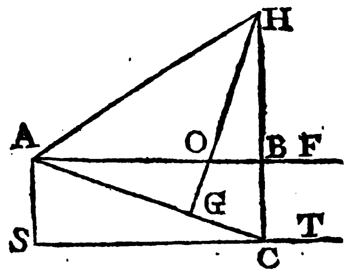
PRO-

PROBLEM LXXXVII. *Answered by Mr.* Perryam.

It is evident, that the given Expression will be a *Maximum*, when $\overline{x)}^{\frac{1}{x}}$, or $\frac{1}{x} \times$ Log. x is a *Maximum*; in which Case its Fluxion $\frac{\dot{x} - \dot{x} \text{ Log. } x}{x^2} = 0$, and Log. $x = 1$, whence $x = 2.718$. Moreover, by putting $y = \overline{x)}^{\frac{1}{x}}$ the Expression will become y^y, which is known to be a *Minimum*, when the Hyp. Log. $y = -1$, or $y = 0.36768$, &c.

PROBLEM LXXXVIII. *Answered by* John Turner.

By *Simpson*'s Laws of Projectiles (N° 486 of the Transactions) Sine of 60° (double Elevation) : Radius : : 1500 (the given Random : 1730.05, the greatest amplitude.



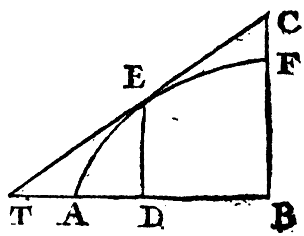
Let SA be the Height of the Peiçe above the Horizon ST, and upon A with a Radius equal to 1774.05, the Sum of the given Height and greatest Amplitude, describe an Arch intersecting the Horizon in C; draw AF parallel, and CB perpendicular to ST; and make CG = SA; also draw GH, perpendicular to AC, intersecting AF in O, and CB (produced) in H; join A, H and BAH will

will be the Elevation required; which will be found to be $44°.17'.24''$.

The Demonstration whereof will appear in the Transactions above quoted.

PROBLEM LXXXIX. *Answered by* John Turner.

Let $TB = b$, $BC = c$, and AD $(= AT)$ $= x$, then $b : c :: 2x : \frac{2cx}{b} = DE$, whence the Parameter



$\left(\frac{DE^2}{AD}\right)$ is $= \frac{4c^2x}{b^2}$ and consequently BF^2 $(= AB \times \textit{Param.}) = \frac{4c^2x}{b^2} \times \overline{b - x} = y^2$; therefore if these Values of the Parameter and Ordinate be substituted in $\left(\frac{\overline{a^2 + 4y^2}|^{\frac{3}{2}}}{a} - a^2\right)$ the general Expression of the Surface, (as found in *Simpson*'s Fluxions) we shall have $\overline{\frac{16c^4x^2}{b^4} + \frac{16c^2bx - 16c^2x^2}{b^2}}|^{\frac{3}{2}} \times \frac{b^2}{4c^2x} - \frac{16c^4x^2}{b^4}$, or (throwing off the constant Quantities, and substituting z for $x^{\frac{1}{2}}$, and d^2 for $c^2 - b^2$) $\overline{dz^4 + b^3z}|^{\frac{3}{2}} - c^3z^6$ for the Value of the Surface, whose Fluxion $\overline{12dz^3 + 3b^3} \times \overline{dz^4 + b^3z}|^{\frac{1}{2}} - 12c^3z^5$

I reduced

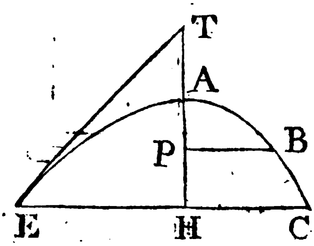
reduced, becomes (by restoring the Value of z) $16\times\overline{d^3-c^6}\times x^3+24\,b^3\,d^2\,x^2+9\,d\,b^6\,x+b^9=0$; whence x may be found.

COROLLARY.

If IB = BC, then $16x^3=b^3$ and $x=\dfrac{b}{2\sqrt[3]{2}}$.

PROBLEM XC. *Answered by* John Turner.

AS ET : EH : : $\sqrt{2}:1::400:282.8426$, the Velocity in the Direction of the Horizon, which



put $=v$, and let a be the Velocity with which the Resistance of the Body is equal to its Gravity; then (by Art. 373. *Simpson's* Fluxions) the Velocity at the Vertex A will be expressed by $\dfrac{av}{\sqrt{a^2+2v^2Q}}$ and the Distance or Arch EA by $\frac{1}{2}d\times$ Hyp. Log. $\overline{1+\dfrac{2v^2Q}{a^2}}$, where Q is equal to $\frac{1}{2}w\sqrt{1+w^2}+\frac{1}{2}\times$ Hyp. Log. $w+\sqrt{1+w^2}$, (w being the Tangent of the given Elevation) and $d=\dfrac{a^2}{32\frac{1}{6}}$; therefore in the present Case $2v^2$ being $=a^2$, and $w=1$, the Velocity at A will become $\dfrac{v}{\sqrt{1+Q}}$ and the Arch

EA

$EA = \frac{a^2}{64\frac{1}{3}} \times$ Hyp. Log. $\overline{1+Q} = 1901.14$.

Let (193) the Velocity at A thus found be denoted by c, then the Velocity at any other Point B, in the Direction PB, will be expressed by $\frac{ac}{\sqrt{a^2+2c^2Q}}$ and the Arch AB by $\frac{1}{2} d \times$ Hyp. Log. $\overline{1+\frac{2c^2Q}{a^2}}$, w being the Tangent of the Angle ABP and Q as above; whence $1 : \sqrt{1+w^2} :: \frac{ac}{\sqrt{a^2+2c^2Q}} : \frac{ac\sqrt{1+w^2}}{\sqrt{a^2+2c^2Q}}$ the absolute Velocity at B, which will be a *Minimum*, when $\frac{a^2+2c^2Q}{1+w^2}$ is a *Maximum*: Therefore its Fluxion $2c^2\dot{Q} \times \overline{1+w^2} - 2w\dot{w} \times \overline{a^2+2c^2Q} = 0$, or $\frac{\overline{1+w^2}|^{\frac{3}{2}}}{w} = \frac{a^2}{c^2} + 2Q = \frac{a^2}{c^2} + w\sqrt{1+w^2}$ + Hyp. Log. $w + \sqrt{1+w^2}$, and consequently $\frac{\sqrt{1+w^2}}{w}$ —Hyp. Log. $w + \sqrt{1+w^2} = \frac{a^2}{c^2} = 4.29$, whence $w = .227 =$ the Tangent of $12° 48'$; therefore the least Velocity is $= 188.24$, the Distance described from the Vertex $= 239.25$, and the whole Distance described $= 2140.30$.

FINIS.

ERRATA in the Differtations.

Diff. 1. p. 2. line 7. for *methematical* read *mathematical*. P. 3. lines 22, 23. for *the univerfe to influence* read *the univerfe and their influence*. P. 16. line 21. for *found* read *compounded*.

Diff. 2. p. 60. lines 10, 11. for $5x^3+2x^2+6x'$ read $5x^2+2x^1+6x^0$.

Diff. 3. p. 130. line 25. for *ao into ao* read *am into am*.

Diff. 4. p. 196. line 3. for *by* read *be*; line 30. for $S=tv$ read $f=tv$; line 33. for *$V=v$ then $T:t$* read *$T=t$, then $V:v$*. line 37. dele *then*; line 38. for *AC* read *Ab*; line 39. dele $=$. P. 198. line 13. for *would* read *might*; line 22. for *as R, S, be the* read *as R, S, be as the*. P. 200. lines 10, 11, 12. dele *Hitherto we have fpoken concerning uniform motions only, when both the generating quantities flow equally*. P. 209. laft line, for *Tringle* read *Triangle*.

Diff. 5. p. 262. line 4. for *where* read *were*. P. 270. line 30. for *evenafcent* read *evanefcent*. P. 273. line 37. for *in Algebra; the Evidence* read *in Algebra; and tho' the Evidence*. P. 274. line 3. for *yet it is* read *yet fince it is*.

Diff. 6. p. 343. line 19. dele *in*. P. 344. l. 10. for *their* read *this*. P. 345. lines 6, 7. for $3x^2\dot{x}+3xo\dot{x}^2+o^2\dot{x}^3-y^2\dot{x}-2xy\dot{y}$ read $3x^2\dot{x}+3xo\dot{x}^2+o^2\dot{x}^3-y^2\dot{x}-2y\dot{y}-2oy\dot{x}\dot{y}-xo\dot{y}^2-o^2\dot{x}\dot{y}^2+aa\dot{z}=o$. P. 354. line 22. for *that we* read *that if we*.

E R R A T A.

In Prob. 19, Page 48, for $\frac{3}{4}$ read $\frac{3}{2}$; Page 169, for m read m^2, and for $4n$ read $4n^2$; Page 174, for the Series in the 10th, 11th, and 12th Lines, read

$$\frac{1}{r}+\frac{2}{r^2}+\frac{3}{r^3}+\frac{4}{r^4}+\frac{5}{r^5}\ \&c.\ \frac{1}{r}+\frac{4}{r^2}+\frac{9}{r^3}+\frac{16}{r^4}+\frac{25}{r^5}\ \&c.\ \text{and}\ \frac{1}{r}+\frac{8}{r^2}+\frac{27}{r^3}+\frac{64}{r^4}+\frac{125}{r^5}$$

&c. respectively. In Problem 59, for 23 read 32. In Problem 60, Line 4, after *sailed*, read *by the Ship*. Page 256, for LVII. read LXVII. and in Line 2d, Prob. 67. after *that* dele *AB*. In Page 269, Line 12 from the Bottom, for *it*, read *the Objection*. Page 306, Line 1, for $+$ read $\times$. In Prob. 68, Page 315, Line 1st, after *AC* read $FG = AB$. In the third Line of the Lemma, Page 326, for *Surface* read *Plane*. In Cor. 3. Page 331. for D^3 read B^3, and for *Distance* (in Line 6) read Distances. In Cor. 9, Page 333, for *accelerating* read *absolute*. In Page 337, Line last, for *Speroid* read *Spheroid*. In Prob. 78. for *of the Base* read *thereof*. After the last Word in Prob. 79. read *and its Area* $= 720$;

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